APOLAR SETS TO REDUCIBLE FORMS

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In this work, we study forms of the type $F=(x_1^2+\cdots+x_m^2)(y_1^2+\cdots+y_n^2)$ for $n\geqslant 2$, $m\geqslant 1$. For $m=1,2$, we provide a complete description of the structure of minimal apolar sets to F. This study enables the determination of the forbidden locus of F and yields the structure of its minimal apolar ideal, including the corresponding Hilbert function. Our approach improves the lower bound and upper bounds of the Waring rank of F in the case $m\geqslant 3$.

Introduction: This work concerns the study of apolar sets associated to forms, a topic situated within the broader context of the **Waring rank problem** for homogeneous polynomials. Let $S=\mathbb{C}[x_0,x_1,\dots,x_n]$ be a polynomial ring in $n+1$ variables over $\mathbb{C}$. Let F be a homogeneous polynomial of degree d in S. Then it is a well known fact that there exists $L_1,L_2,\dots,L_l\in S_1$, and $\alpha_1,\alpha_2,\dots,\alpha_l\in k$ such that $F=\sum_{i=1}^{l}\alpha_i {L_i}^d$. This is a **Waring decomposition** of F. The least integer r such that F has a Waring decomposition F with exactly r summand is the **Waring rank** of F, $\mathrm{rk}(F)$.
Determining or even bounding the Waring rank of a given form is a fundamental but notoriously difficult problem in algebraic geometry and computational algebra. One powerful tool for studying Waring rank is apolarity theory.
Let $T=\mathbb{C}[X_0,X_1,\dots,X_n]$ be the ring of differential operators with the natural action on S defined by $X_i=\frac{\partial}{\partial x_i}$. For a homogeneous form F, the apolar ideal of F is defined as $F^\perp=\{g\in T\mid g\circ F=0\}$. This is a homogeneous Artinian Gorenstein ideal. A set of points $\mathbb{X}\subseteq\mathbb{P}^n$ is said to be **apolar** to F if the defining ideal of $\mathbb{X}$, $I(\mathbb{X})\subset F^\perp$. The **Waring locus** (see [2]) of a form F is given by $\mathcal{W}_F=\{P\in\mathbb{P}^n\mid P\in\mathbb{X},\ I_{\mathbb{X}}\subset F^\perp,\ |\,\mathbb{X}\,|=\mathrm{rk}(F)\}$. The **locus of forbidden points** $\mathcal{F}_F$ of a form is defined as complement of $\mathcal{W}_F$, i.e. $\mathcal{F}_F=\mathbb{P}^n\setminus\mathcal{W}_F$.

Theorem. Let $F=(x_1^2+\cdots+x_m^2)(y_1^2+\cdots+y_n^2)$, $n\geqslant m\geqslant 2$. Then $n(m+2)\leqslant \mathrm{rk}(F)\leqslant 2mn$. If $m=2$, then the equality holds. If $m\geqslant 3$, then the lower bound is strict.

Proof. Idea: We use e-computability (see [1] to obtain lower bound for $\mathrm{rk}(F)$. We obtain an upper bound by apolarity lemma, where we give an apolar ideal of points in $F^\perp$. For $m\geqslant 3$, we improve the lower bound by studying the structure of minimal apolar subsets of F. □

Theorem. Let $F=x^2(y_1^2+\cdots+y_n^2)$ with $n\geqslant 2$ and let $\mathbb{X}$ be a set of $\mathrm{rk}\,F=3n$ distinct points apolar to F. Then the following facts hold:

i) The forbidden locus of F is $\mathcal{F}_F=V(Y_1^2+\cdots+Y_n^2)$.

ii) If $\mathbb{Y}$ is the projection of $\mathbb{X}$ from $V(Y_1,\dots,Y_n)$ on $V(X)$, then $\mathbb{Y}$ is a set of n distinct points in linear general position, i.e. the Hilbert function of $\mathbb{Y}$ is

i	0	1	2
$\mathrm{HF}(\mathbb{Y},i)$	1	n	n

and $\mathrm{HF}(\mathbb{Y},i)=n$ for $i\geqslant 3$.

iii) Let $\mathbb{Y}=\{P_1,\dots,P_n\}$. If $\Lambda_i=\langle P_i,P\rangle$, where $\langle P_i,P\rangle$ denotes the linear span of P_i and $P=[1:0:0:\cdots:0]$, then $\mathbb{X}\cap\Lambda_i$ is a set of three distinct points for $0\leqslant i\leqslant n$.

iv) The Hilbert function of $\mathbb{X}$ is

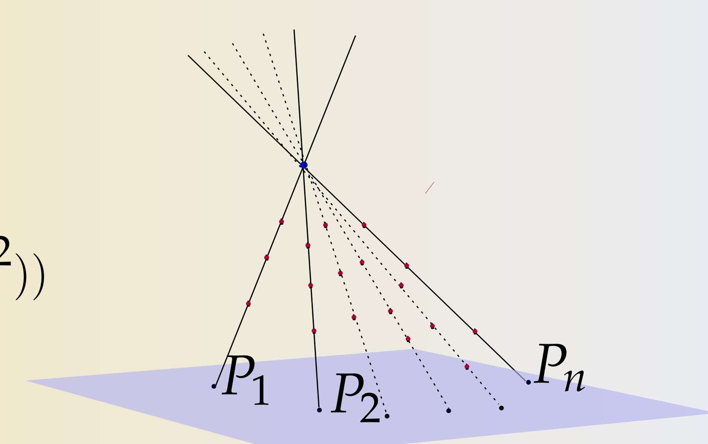
i	0	1	2	3
$\mathrm{HF}(\mathbb{X},i)$	1	n	$2n+1$	$3n$

and $\mathrm{HF}(\mathbb{X},i)=3n$ for $i\geqslant 3$.

v) The ideal of $\mathbb{X}$ up to change of coordinates is given by

$$I(\mathbb{X})=(\{Y_iY_j\}_{1\leqslant i<j\leqslant n},\ X^3+\sum_{i=2}^{n}L_i(Y_1^2-Y_i^2))$$

for some linear forms $L_i\in\mathbb{C}[X,Y_1,\dots,Y_n]$ for $2\leqslant i\leqslant n$.



Theorem. Let $F=x_1x_2(y_1^2+\cdots+y_n^2)$ and let $\mathbb{X}$ be a set of $\mathrm{rk}\,F=4n$ distinct points apolar to F. Then the following holds true.

i) The forbidden locus of F is $\mathcal{F}_F=V(X_1X_2(Y_1^2+\cdots+Y_n^2))$.

ii) If $\mathbb{Y}$ is the projection of $\mathbb{X}$ from $V(Y_1,\dots,Y_n)$ on $V(X_1,X_2)$, then $\mathbb{Y}$ is a set of n distinct points in linear general position, that is

i	0	1	2
$\mathrm{HF}(\mathbb{Y},i)$	1	n	n

and $\mathrm{HF}(\mathbb{Y},i)=n$ for $i\geqslant 3$.

iii) Let $\mathbb{Y}=\{P_1,\dots,P_n\}$ and let $\ell=V(Y_1,\dots,Y_n)$. If $\Lambda_i=\langle\ell,P_i\rangle$, then $\mathbb{X}\cap\Lambda_i$ is a set of four distinct points for $1\leqslant i\leqslant n$ contained in two lines passing through P_i.

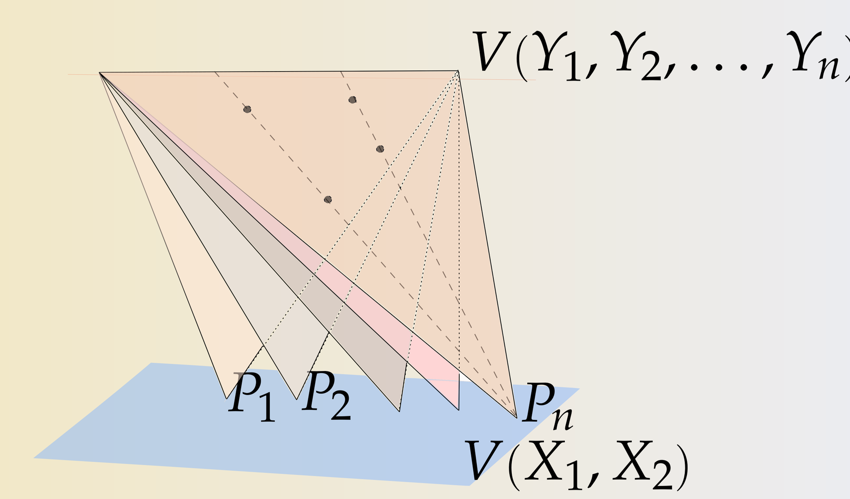
iv) the Hilbert function of any minimal apolar set $\mathbb{X}$ is

i	0	1	2	3
$\mathrm{HF}(\mathbb{X},i)$	1	n+2	3n+1	4n

and $\mathrm{HF}(\mathbb{X},i)=4n$ for all $i\geqslant 4$.

v) The ideal of $\mathbb{X}$ up to change of coordinates is given by

$$I(\mathbb{X})=(\{Y_iY_j\}_{1\leqslant i<j\leqslant n},\mathcal{Q}_1,\mathcal{Q}_2)$$



where

$\mathcal{Q}_1=X_1^2+\sum_{i=2}^{n}\alpha_i(Y_1^2-Y_i^2)$, $\mathcal{Q}_2=X_2^2+\sum_{i=2}^{n}\beta_i(Y_1^2-Y_i^2)$ **for $2\leqslant i\leqslant n$, $\alpha_i,\beta_i\in\mathbb{C}$**

vi) If $\mathbb{W}$ is the projection of $\mathbb{X}$ from $V(X_1,X_2)$ on $V(Y_1,\dots,Y_n)$, then $\mathbb{W}$ is a set of at most $2n$ points. Moreover, $\mathbb{W}$ always has an even number of points.

One of the key ideas in our results is to bound the Hilbert function of the apolar ideal of points in a certain degree, and hence gather more information on the projection. Moreover, nice structure of $F^\perp$ is an added bonus, makes a lot of computations easier.

References

[1] E. Carlini, M. V. Catalisano, L. Chiantini, A. V. Geramita, and Y. Woo. Symmetric tensors: rank, Strassen's conjecture and e-computability. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(1):363–390, 2018.

[2] E. Carlini, M. V. Catalisano, and A. Oneto. Waring loci and the Strassen conjecture. *Adv. Math.*, 314:630–662, 2017.

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