

Hilbert functions in Waring decomposition and singularity theory

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by

Meghana Bhat
(Roll No. MA21DPF01)

Under the guidance of
Prof. Shreedevi K. Masuti



DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DHARWAD

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To my parents and family

Abstract

This thesis focuses on the study of the Hilbert functions and its applications to Waring decomposition of homogeneous polynomials (forms) and singularity theory. Chapters 2 and 3 are devoted to the study of Waring decomposition and computation of rank of special binomials and more forms. In Chapters 4 and 5 we give applications of Hilbert functions in singularity theory. We characterize Gorensteinness of associated graded rings using Hilbert functions.

Chapter 2 is motivated by the question of understanding the minimal apolar sets of homogeneous polynomials that arise in the Waring decomposition. In particular, we study forms of the type $F = (x_1^2 + \cdots + x_m^2)(y_1^2 + \cdots + y_n^2)$. For $m = 1, m = 2$, and for any n , using projections we describe: the forbidden locus, the structure, and the Hilbert function of all minimal apolar sets. In particular, we show that every minimal apolar ideal has the same Hilbert function. Moreover, we obtain that all defining ideals have the same canonical form up to the orthogonal action. For $m, n \geq 3$, we provide new lower and upper bounds for the Waring rank.

In Chapter 3 our goal is to understand the Waring rank of certain families of forms. First, we give a formula for the Waring rank of forms whose annihilator ideal is of the type $(X_0^{a_0} + G, X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n})$ where G is a homogeneous form of degree a_0 with $\deg_{X_0} G < a_0$. We call such ideals quasi-monomial complete intersections. As an application, we compute the Waring rank of all binomials whose annihilator ideal is a complete intersection.

In Chapter 4 we present our study on symmetric decomposition of the Hilbert function of an ideal in an Artinian Gorenstein (AG) local k -algebra A . This is motivated from the work of A. Iarrobino on the symmetric decomposition of the Hilbert function of A . This became a very powerful tool for classifying the Hilbert functions of AG k -algebras. We introduce the symmetric decomposition of the Hilbert function of any ideal I in A . Using this we characterize Gorensteinness of $G(I)$. Our hope is that this result will be useful in classifying the possible Hilbert function of an ideal in an AG k -algebra. We illustrate this by giving a complete list of 2-admissible sequences of length at most 3 and with $h_0 = 2$ that are realizable by an ideal in an

AG k -algebra.

Chapter 5 is devoted to understand Gorensteinness of associated graded rings of a filtration $\mathcal{F}$ in a Gorenstein local ring. We give a criterion for Gorensteinness of the associated graded ring of $\mathcal{F}$ in terms of the Hilbert coefficients of $\mathcal{F}$ in some cases. Further, we present ring-theoretic properties of the normal tangent cone of the maximal ideal of $A = S/(f)$ where $S = K[[x, y_1, \dots, y_m]]$ is a formal power series ring over an algebraically closed field K , and $f = x^a - g(y_1, \dots, y_m)$, where g is a polynomial with $g \in (y_1, \dots, y_m)^b \setminus (y_1, \dots, y_m)^{b+1}$, and a, b, m are integers. We show that the normal tangent cone $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay if $a \leq b$ and $a \neq 0$ in K . Moreover, we give a criterion of the Gorensteinness of $\overline{G}(\mathfrak{m})$.

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Notation

k	a field.
$\text{Rk } F$	Waring rank of F .
$\mathcal{W}_F$	Waring locus of F .
$\mathcal{F}_F$	Forbidden locus of F .
$I(\mathbb{X})$	defining ideal of the variety $\mathbb{X} \subseteq \mathbb{P}^n$.
$\text{HF}(\mathbb{X}, i)$	Hilbert function of $T/I(\mathbb{X})$ at i .
$F^\perp$	annihilator ideal/perp ideal.
$\text{Sing}(\mathbb{V})$	Singular locus of the variety $\mathbb{V}$.
$V^\vee$	$\text{Hom}_k(V, k)$.
$G(I)$	associated graded ring of an ideal $I \subset A$.
$\mathcal{F}$	Hilbert filtration.
$G(\mathcal{F})$	associated graded ring of a filtration $\mathcal{F}$.
$HS_{G(\mathcal{F})}(z)$	Hilbert series of $G(\mathcal{F})$ in the variable z .
$r_Q(\mathcal{F})$	reduction number of $\mathcal{F}$ with respect to Q .
$r(\mathcal{F})$	reduction number of $\mathcal{F}$.
$\text{nr}_Q(\mathcal{F})$	relative reduction number of $\mathcal{F}$ with respect to Q .
$\text{nr}(\mathcal{F})$	relative reduction number of $\mathcal{F}$.

शनैः पन्थाः शनैः कन्था शनैः पर्वतमस्तके ।
शनैः विद्या शनैः वित्तं पञ्चैतानि शनैः शनैः ॥

- सुभाषितम्

“Mathematics is the music of reason”

- J. J. Sylvester

“Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.”

- Hermann Weyl

Chapter 1

Introduction

The theme of this thesis is Hilbert functions. We focus on its applications to Waring decomposition and singularity theory.

Commutative algebra is a tool for studying polynomials and the algebraic structures derived from them. One of the foundational developments comes from the work on polynomials, invariants, and syzygies by Hilbert in 1890. In [32] Hilbert introduced what is now called the *Hilbert function*, an invariant that measures the size of the graded pieces of a graded algebra. Although defined in elementary linear-algebraic terms, the Hilbert function encodes deep geometric and algebraic information. In the projective setting, it determines important discrete invariants of a projective variety, including its dimension, arithmetic genus to mention a few. More generally, Hilbert function has become a fundamental tool in commutative algebra and algebraic geometry, with applications in areas ranging from the study of syzygies and resolutions to tensor decomposition, singularity theory, and combinatorics.

Let $T = k[X_0, X_1, \dots, X_n]$ be a standard graded polynomial ring over a field k , and let $I \subseteq T$ be a homogeneous ideal. Then $R = T/I$ is a standard graded k -algebra. The *Hilbert function* of R is defined by

$$\mathrm{HF}(R, i) = \dim_k R_i = \dim_k (T/I)_i, \text{ for } i \geq 0.$$

Thus, $\mathrm{HF}(R, i)$ records the dimension of the degree i component of R . Equivalently, it measures the number of linearly independent forms of degree i modulo the relations imposed by the ideal I . For basic properties of $\mathrm{HF}(R, i)$, we refer to [7].

One of the reasons Hilbert functions are so useful is that they compress subtle algebraic information into a numerical profile. In the Artinian case, the Hilbert function is a finite se-

quence, and its shape is constrained by strong algebraic conditions. For Artinian Gorenstein algebras, for example, the Hilbert function often reflects duality and symmetry phenomena.

Hilbert functions also arise naturally in singularity theory. If one studies the local ring of a singular point on a variety, then the Hilbert function of the local ring and of its associated graded ring captures important information about the singular points. In particular, the Hilbert - Samuel function and the corresponding Hilbert–Samuel polynomial encode the multiplicity of the singularity, one of its most fundamental numerical invariants. The associated graded ring with respect to the maximal ideal encodes the tangent cone of the singularity, and its Hilbert function provides numerical information about the local structure of the singular point through the dimensions of the quotients $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. In the study of Artinian Gorenstein local algebras, Hilbert functions place strong constraints on the possible local algebra structures and therefore on the singularities they represent. Thus, in both projective and local settings, Hilbert functions provide a unifying numerical invariant through which one can approach questions of geometry, algebra, and singularities.

In this thesis, we present two problems of different flavors: Waring decomposition and problems in singularity theory: symmetric decomposition of the Hilbert function of an ideal and Gorensteinness of associated graded rings in terms of Hilbert coefficients, where the understanding of Hilbert functions plays a great role.

The first part of the thesis concerns the Waring rank problem for forms, where Hilbert functions of apolar algebras and characterization of Hilbert functions of ideals defining distinct points in projective space is very useful.

Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the standard graded polynomial ring over $\mathbb{C}$, and let $F \in S$ be a homogeneous polynomial of degree d , also known as a degree d form. It is well known that [39] there exist $L_1, \dots, L_r \in S_1$ such that

$$F = \sum_{i=1}^r \alpha_i L_i^d,$$

for some $\alpha_i \in \mathbb{C}$, $1 \leq i \leq r$. Such a decomposition of F is called a *Waring decomposition* or *sum of powers decomposition*. The least integer r such that there exists a Waring decomposition of F with exactly r summand is called the *Waring rank* of F , denoted by $\text{Rk } F$.

Question A. Given $F \in S_d$, compute the Waring rank and a Waring decomposition of F .

This simple looking problem is extremely difficult to solve. Formal hardness is proved in [68]. It is a special case of tensor decomposition, since homogeneous polynomials may be

identified with symmetric tensors and Waring decompositions correspond to decompositions into symmetric rank-one tensors. Because of this, the problem is relevant not only in commutative algebra and algebraic geometry, but also in areas such as algebraic statistics, signal processing [44], quantum information theory [3, 10], and complexity theory [45–47].

One of the landmark results in the Waring rank problems is due to J. Alexander and A. Hirschowitz, where they determined the rank of a generic form [1]. However, despite our understanding of the generic case, computing the Waring rank of a specific polynomial remains a challenging open problem; no efficient algorithm is currently known.

Significant progress has been made for specific families of polynomials. For example, the Waring rank of binary forms was classically known due to Sylvester. In [69, 70] an algorithm to compute the Waring rank of binary forms is presented. Similarly, the Waring rank of a quadratic form in any number of variables is classically known, as it coincides with the rank of the associated symmetric matrix.

In more recent years, the rank has been determined for several other families: The Waring rank of monomials is obtained in [11, 13]. The Waring rank of binary binomials, and of elementary symmetric polynomials are computed in [8] and [48], respectively. The study of reducible forms has also been a subject of long-standing interest due to their importance in analysis. While the rank of a quadratic form itself is elementary, the ranks of its powers are far more intricate and have been studied extensively in [25, 61]. The Waring rank of reducible cubic forms was specifically investigated in [16]. More recently, the rank of broader classes of reducible forms has been explored in [18].

Given a linear form L appearing in a Waring decomposition of F , one naturally associates it with a point of $\mathbb{P}^n$. With this connection, Apolarity Lemma 2.1.1 provides a beautiful link between Waring decompositions and projective geometry. Nevertheless, the structure of the point configurations arising from minimal decompositions is still far from being understood. Even when the numerical value of the rank is known, the corresponding minimal apolar sets often remain geometrically mysterious. Understanding these sets more deeply may lead to new insight into the Waring rank.

This motivates the following question:

Question B. *Given $F \in S_d$, describe algebraic and geometric properties of the minimal apolar sets of F .*

For instance, in [11], the authors proved that every minimal apolar scheme to a monomial

is a complete intersection. A systematic investigation of the structure of minimal apolar sets was initiated in [14], where they introduced the concept of Waring loci of F - a set of all linear forms that appear in some minimal Waring decomposition of F , and forbidden loci - a complement of the set Waring loci in $\mathbb{P}^n$. In this thesis, we continue this line of inquiry by focusing on products of two quadratic forms.

Our main results of Chapter 2 are summarized below:

Theorem I. A. *Let $F = x^2(y_1^2 + \cdots + y_n^2)$ with $n \geq 2$. If $\mathbb{X}$ is a set of $\text{Rk } F = 3n$ distinct points apolar to F , then the following facts hold:*

i) *The forbidden locus of F is*

$$\mathcal{F}_F = V(Y_1^2 + \cdots + Y_n^2).$$

ii) *If $\mathbb{Y}$ is the projection of $\mathbb{X}$ from $V(Y_1, \dots, Y_n)$ on $V(X)$, then $\mathbb{Y}$ is a set of n distinct points in linear general position, that is*

$$\begin{array}{c|c|c|c} i & 0 & 1 & 2 \\ \hline \text{HF}(\mathbb{Y}, i) & 1 & n & n \end{array},$$

and $\text{HF}(\mathbb{Y}, i) = n$ for $i \geq 3$.

iii) *Let $\mathbb{Y} = \{Q_1, \dots, Q_n\}$. If $\Lambda_i = \langle Q_i, P \rangle$, where $\langle Q_i, P \rangle$ denotes the linear span of Q_i and $P = [1 : 0 : 0 : \cdots : 0]$, then $\mathbb{X} \cap \Lambda_i$ is a set of three distinct points for $1 \leq i \leq n$.*

iv) *The Hilbert function of $\mathbb{X}$ is*

$$\begin{array}{c|c|c|c|c} i & 0 & 1 & 2 & 3 \\ \hline \text{HF}(\mathbb{X}, i) & 1 & n & 2n + 1 & 3n \end{array},$$

and $\text{HF}(\mathbb{X}, i) = 3n$ for $i \geq 3$.

v) *Up to change of coordinates, the ideal of $\mathbb{X}$ is*

$$I(\mathbb{X}) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X^3 + \sum_{i=2}^n L_i(Y_1^2 - Y_i^2))$$

for linear forms $L_i \in \mathbb{C}[X, Y_1, \dots, Y_n]$, $2 \leq i \leq n$.

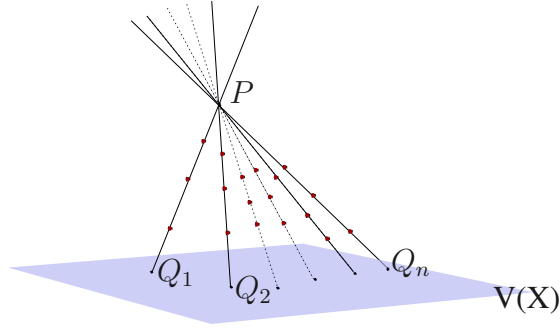


Figure 1.1: A minimal apolar set of $F = x^2(y_1^2 + \cdots + y_n^2)$: red dots represent apolar points contained in n -lines with 3 points on each line.

Our proof techniques make use of the Hilbert function of the annihilator ideal $F^\perp$ and the Apolarity Lemma [2.1.1]. Note that for $F = x^2(y_1^2 + \cdots + y_n^2)$, the Waring rank was known [12, Proposition 4.4]. In the case where $F = (x_1^2 + x_2^2)(y_1^2 + \cdots + y_n^2)$, we also calculate the Waring rank. The second main result in Chapter 2 is the following:

Theorem I. B. *Let $F = (x_1^2 + x_2^2)(y_1^2 + \cdots + y_n^2)$ and $n \geq 2$, then $\text{Rk } F = 4n$. If $\mathbb{X}$ is a set of $4n$ distinct points apolar to F , then the following facts hold:*

i) *The forbidden locus of F is given by*

$$\mathcal{F}_F = V(X_1^2 + X_2^2) \cup V(Y_1^2 + \cdots + Y_n^2).$$

ii) *If $\mathbb{Y}$ is the projection of $\mathbb{X}$ from $V(Y_1, \dots, Y_n)$ on $V(X_1, X_2)$, then $\mathbb{Y}$ is a set of n distinct points in linear general position, that is*

$$\begin{array}{c|c|c|c} i & 0 & 1 & 2 \\ \hline \text{HF}(\mathbb{Y}, i) & 1 & n & n \end{array},$$

and $\text{HF}(\mathbb{Y}, i) = n$ for $i \geq 3$.

iii) *Let $\mathbb{Y} = \{Q_1, \dots, Q_n\}$ and let $\ell = V(Y_1, \dots, Y_n)$. If $\Lambda_i = \langle \ell, Q_i \rangle$, where $\langle \ell, Q_i \rangle$ denotes the linear span of ℓ and Q_i , then $\mathbb{X} \cap \Lambda_i$ is a set of four distinct points for $1 \leq i \leq n$ contained in two lines passing through Q_i .*

iv) *Up to change of coordinates, the ideal of $\mathbb{X}$ is*

$$I(\mathbb{X}) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, Q_1, Q_2)$$

where $Q_1 = X_1^2 + \sum_{i=2}^n \alpha_i(Y_1^2 - Y_i^2)$, $Q_2 = X_2^2 + \sum_{i=2}^n \beta_i(Y_1^2 - Y_i^2)$ for some $\alpha_i, \beta_i \in \mathbb{C} \setminus \{0\}$, $2 \leq i \leq n$ and $\sum_{i=2}^n \alpha_i \neq 0$, $\sum_{i=2}^n \beta_i \neq 0$.

v) If $\mathbb{W}$ is the projection of $\mathbb{X}$ from $V(X_1, X_2)$ on $V(Y_1, \dots, Y_n)$, then $\mathbb{W}$ is a set of $2k$ points with $k \leq n$.

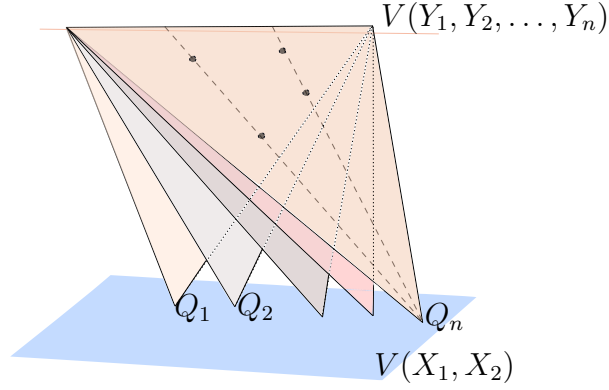


Figure 1.2: A minimal apolar set of $F = (x_1^2 + x_2^2)(y_1^2 + \dots + y_n^2)$: black dots represent minimal apolar points contained in n -planes with 4 points on each plane.

Further, we obtain new bounds on the Waring rank of the product of Fermat squares.

Theorem I. C. *Let $n \geq m \geq 3$. If we let $F = (x_1^2 + \dots + x_m^2)(y_1^2 + \dots + y_n^2)$, then*

$$n(m+2) < \text{Rk } F \leq 2mn.$$

In Chapter 3 our focus is on computing the Waring rank of special forms whose annihilator ideal is a quasi-monomial complete intersection. One of the simplest forms after monomials is binomials. In [9], the Waring rank of binary binomials is computed. Recently, the Waring decomposition of special binomials in [17] is studied motivated by their connections with complexity theory. Our goal is to compute the Waring rank of binomials inspired by recent works [2, 21, 67], where the authors characterized binomials F such that the corresponding apolar algebra $T/F^\perp$ is a complete intersection, where $T = \mathbb{C}[X_0, \dots, X_n]$ and $F^\perp$ is a set of all partial derivatives that kill F . In fact, they showed that the annihilator ideal $F^\perp$ of such a binomial F is a *quasi-monomial complete intersection* in “most” of the cases, that is, an ideal of the form

$$F^\perp = (X_0^{a_0} + G, X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n})$$

where G is a homogeneous form in T of degree a_0 with $\deg_{X_0} G < a_0$. Motivated by this, more generally, we ask:

Question C. *Compute the Waring rank of a form $F \in S_d$ whose annihilator ideal $F^\perp \subseteq T$ is a complete intersection of the form $(X_0^{a_0} + G_0, X_1^{a_1} + G_1, \dots, X_n^{a_n} + G_n)$ where $\deg G_i = a_i$ and $\deg_{X_i} G_i < a_i$ for $0 \leq i \leq n$.*

We answer Question [C](#) when $G_1 = \dots = G_n = 0$. Our proof techniques are motivated by the computation of the Waring rank of monomials. For the case of quasi-monomials, we first employ e-compatibility to establish a definitive lower bound on the rank. We then prove this bound is sharp by constructing a set of distinct points within $F^\perp$ whose cardinality matches this bound. By the Apolarity lemma, we achieve the rank in this case. Our result is summarized below:

Theorem II. A. *Let $F \in T$ be such that $F^\perp = (X_0^{a_0} + G, X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n})$ where G is a form of degree a_0 with $\deg_{X_0} G < a_0$ and $a_1 \leq a_2 \leq \dots \leq a_n$. Then*

$$\text{Rk } F = \begin{cases} a_0 \prod_{j=2}^n a_j & \text{if } a_1 \leq a_0 \text{ or} \\ & a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ does not have a multiple factor.} \\ \prod_{j=1}^n a_j & \text{if } a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ has a multiple factor.} \end{cases}$$

Using a description of the generators of binary binomial $F^\perp$ given by Shibata [\[67\]](#), Theorem 1.3], we reprove the Waring rank of binary binomials proved by Brustenga i Moncusí, and Masuti in [\[8\]](#), Theorem 3.1].

Theorem II. B. *Let $F = x_0^{a_0} x_1^{a_1} (x_0^{b_0} - x_1^{b_0})$ be such that $b_0, a_0, a_1 \in \mathbb{N}$ with $b_0 \geq 1, 0 \leq a_0 \leq a_1$. Let $v = \min\{i \mid a_0 + 1 \leq ib_0\}$, $w = \min\{i \mid a_1 + 1 \leq ib_0\}$. Then we have the following:*

$v < w$	$a_1 + 1 < a_0 + b_0 + 1$ and $a_1 + 1 - vb_0 \geq 2$	$\text{Rk } F = a_0 + b_0 + 1$
	<i>Otherwise</i>	$\text{Rk } F = a_1 + 1.$
$v = w$	$vb_0 \geq a_0 + a_1 + 2 - (v - 1)b_0, a_1 + 1 - (v - 1)b_0 \leq 1$	$\text{Rk } F = a_0 + a_1 + 2 - (v - 1)b_0$
	<i>Otherwise</i>	$\text{Rk } F = vb_0.$

We now present our results on the Waring rank of binomials whose annihilator ideal is a complete intersection. We note that by the results of [2, 21, 67] such $F^\perp$ in most cases is a quasi-monomial complete intersection. But to apply the result of quasi-monomial complete intersection in the case of the Waring rank of binomials, we need to know whether a certain generator of the annihilator ideal is square-free, which in general is not an easy task. In the rest of the cases, we use a different method to compute the rank.

Note that if F is a binary binomial, then $F^\perp$ is a complete intersection, since Artinian Gorenstein ideals in codimension two are complete intersections. We first present the case when $F = F_1 F_2$ and F_1 is a binary binomial and F_2 is a monomial.

Theorem II. C. *Let $F = F_1 F_2$ where $F_1 = x_0^{a_0} x_1^{a_1} (x_0^{b_0} - x_1^{b_0})$, and $F_2 = x_2^{a_2} \cdots x_n^{a_n}$ such that $a_0 \leq a_1$, and $a_2 \leq \cdots \leq a_n$. Let $\lambda = \min\{a_2 + 1, a_0 + b_0 + 1\}$, and*

$$v = \min\{i \mid a_0 + 1 \leq i b_0\}, \quad w = \min\{i \mid a_1 + 1 \leq i b_0\}.$$

If $v < w$, then

$$\text{Rk } F = \begin{cases} \frac{(a_0 + b_0 + 1)}{\lambda} \prod_{i=1}^n (a_i + 1) & \text{if } \lambda \leq a_1 + 1 \text{ or } (a_1 + 1 < \lambda \text{ and } a_1 + 1 - v b_0 \leq 1) \\ (a_0 + b_0 + 1) \prod_{i=2}^n (a_i + 1) & \text{otherwise.} \end{cases}$$

If $v = w$, then we have the following cases:

$$\text{Rk } F = \begin{cases} \deg P \deg Q \left(\prod_{i=3}^n (a_i + 1) \right) & \text{if one of the following holds:} \\ & (i) \ a_2 + 1 \leq \min\{\deg P, \deg Q\} \\ & (ii) \ \deg P \leq a_2 + 1 \leq \deg Q \\ & (iii) \ \deg Q \leq a_2 + 1 \leq \deg P \text{ and } Q \text{ is square-free} \\ \text{Rk } F_1 \left(\prod_{i=2}^n (a_i + 1) \right) & \text{otherwise,} \end{cases}$$

$$\text{where } F^\perp = \left(P = \sum_{i=0}^v X_0^{i b_0} X_1^{(v-i) b_0}, Q = \sum_{i=0}^{v-1} X_0^{a_0+1-(v-1-i) b_0} X_1^{a_1+1-i b_0} \right).$$

The next case we consider $F = F_1 F_2$ such that F_1 is a binomial in at least three variables such that $F_1^\perp$ is a complete intersection and F_2 is a monomial.

Theorem II. D. Let $F = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} (x_0^{b_0} - x_1^{b_1} \cdots x_n^{b_n})$. Write $F = F_1 F_2$ where $F_1 = x_0^{a_0} x_1^{a_1} \cdots x_k^{a_k} (x_0^{b_0} - x_1^{b_1} \cdots x_k^{b_k})$ a binomial with $b_i \neq 0$ for $1 \leq i \leq k$, $b_{k+1} = \cdots = b_n = 0$, $F_1^\perp$ is a quasi-monomial complete intersection and $F_2 = x_{k+1}^{a_{k+1}} \cdots x_n^{a_n}$ for some $1 \leq k \leq n$, and $a_{k+1} \leq \cdots \leq a_n$.

If $2 \leq k \leq n$, then

$$\text{Rk } F = \begin{cases} \frac{(a_0 + 1)}{\lambda} \prod_{i=1}^k (a_i + b_i + 1) \prod_{i=k+1}^n (a_i + 1) & \text{if } \lambda \leq a_0 + 1 \\ \prod_{i=1}^k (a_i + b_i + 1) \prod_{i=k+1}^n (a_i + 1) & \text{otherwise} \end{cases}$$

where $\lambda = \min(\{a_i + b_i + 1 \mid 1 \leq i \leq k\} \cup \{a_{k+1} + 1\})$.

Note that using the characterization of [21, 67], the above two theorems exhaust all the binomials whose annihilator ideal is a complete intersection.

In Chapter 4 we prove symmetric decomposition of Hilbert function of an ideal in an Artinian Gorenstein (AG) local ring. Let $(R, \mathcal{M})$ be a local ring over a field k where $k = R/\mathcal{M}$ and J an ideal in R such that $A = R/J$ is an AG k -algebra. One of the most important numerical invariants associated with an AG k -algebra is its Hilbert function. Classifying the Hilbert function of AG k -algebras is a widely open problem in commutative algebra. They have been characterized in certain cases, see [6, 37, 43, 49, 54]. In this section, our results are in the direction of understanding the structure of the Hilbert function of I , namely, the Hilbert function of its associated graded ring $G(I) = \bigoplus_{n \geq 0} I^n / I^{n+1}$ for any ideal I in A . The associated graded ring $G(I)$ plays an important role in algebraic geometry, in particular in the resolution of singularities. There has been a lot of research in the literature on studying the properties of $G(I)$ in terms of the properties of I and a local ring A of any dimension (not just AG k -algebra), see [19, 27, 34, 59, 62–64, 71].

For a standard graded S_0 -algebra where S_0 is an Artinian ring, $S = \bigoplus_{i \geq 0} S_i$ the Hilbert function $\text{HF} : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ is defined as $\text{HF}(S)_i = \ell_{S_0}(S_i)$ where $\ell_{S_0}(S_i)$ denotes the length of S_i as S_0 -module. It is well known that the Hilbert function of a graded AG k -algebra is symmetric. This is no longer true with the Hilbert function of a local AG k -algebra. However, A. Iarrobino in [38] developed a beautiful theory that shows the Hilbert function of an AG k -algebra admits a symmetric decomposition, namely,

$$\text{HF}(G(\mathfrak{m}))_i = \sum_{a=0}^{j-2} H(a)_i,$$

where $H(a)$ is symmetric about $(j - a)/2$. Here j is the *socle degree* of A , that is the, maximum integer such that $\mathfrak{m}^j \neq 0$. This became a very powerful tool in classifying the Hilbert functions of local AG k -algebras (see, for example [54]). Recently, there has been intense research on this symmetric decomposition, for instance in [41], the authors have studied the possible vectors $H(a)$ in a symmetric decomposition of the Hilbert function of AG k -algebra, and in [40], the authors have studied its interaction with Jordan type. Very recently, M. Wojtala has extended the symmetric decomposition theory to self-dual modules in [72].

We introduce a symmetric decomposition of the Hilbert function of an arbitrary ideal in an AG local k -algebra. We prove that:

Theorem III. A. *Let $(A, \mathfrak{m})$ be an AG k -algebra, and I an ideal in A such that $I^u \neq 0$ and $I^{u+1} = 0$. Then the Hilbert function of I has a decomposition*

$$\text{HF}(G(I))_i = \sum_{a=0}^u H(a)_i,$$

where $H(a)$ is symmetric about $(u - a)/2$.

As a consequence, we recover a result by W. Heinzer, M.-K. Kim, and B. Ulrich in [31], which asserts that $G(I)$ is Gorenstein if and only if the Hilbert function of $G(I)$ is symmetric.

Our hope is that Theorem III. A will be useful in classifying the Hilbert function of an ideal in an AG k -algebra. One of the difficulties in understanding the structure of the Hilbert function of $G(I)$ is that, in this case $G(I)_0 = A/I$, which is not a field unless $I = \mathfrak{m}$. In [5] C. Blancafort and S. Nollet characterized the Hilbert functions of standard graded algebras of the form $R_0[x_1, \dots, x_b]/I$ where R_0 is an Artinian ring. They call such a sequence a *b-admissible sequence*. This result generalizes a result of Macaulay which characterizes the Hilbert functions of standard graded algebras over a field [50]. We ask:

Question D. *Which b-admissible sequences occur as the Hilbert function of $G(I)$ for some ideal I in an AG k -algebra A ?*

We call a sequence that occurs as the Hilbert function of $G(I)$ for some ideal I in an AG k -algebra A as an *I-Gorenstein sequence*. Note that a Gorenstein sequence refers to a sequence that occurs as the Hilbert function of some AG k -algebra. To the best of our knowledge, this is the first time realizable Gorenstein sequences are studied in this generality. As with the case of Gorenstein sequences, the classification of *b*-admissible sequences that are *I*-Gorenstein is a difficult problem. We are far from having a complete solution. However, we want to investigate

the problem in certain cases. For instance, the structure of the Hilbert function of AG k -algebra of codimension two is well known [49, 51]. Inspired by this, we investigate the structure of the Hilbert function of an ideal in a codimension two AG k -algebra.

In Chapter 5 we study the Gorenstein property of associated graded rings arising from filtrations of $\mathfrak{m}$ -primary ideals. These graded rings encode asymptotic information about the filtration and have been studied extensively as a bridge between local algebra and the geometry of singularities. In particular, their homological properties, especially the Cohen–Macaulayness and Gorensteinness, often provide information of the underlying local ring and its ideals.

In Chapter 5, our study shows that the Gorensteinness of an associated graded ring can often be detected through Hilbert coefficients 5.0.1. The Hilbert function of a filtration agrees for large values with a polynomial, and the coefficients of this polynomial carry important numerical information about the filtration. These coefficients have long been used to study depth, multiplicity, and reduction numbers. We show that they also provide effective criteria for determining when the associated graded ring is Gorenstein.

Our motivation comes in part from two important classes of filtrations: the ordinary I -adic filtration $\{I^n\}_{n \in \mathbb{Z}}$ and the integral closure filtration of ideals. The latter is particularly significant in the study of singularities, where integral closure and normalization play a fundamental role. Even when the local ring is Cohen–Macaulay or Gorenstein, the corresponding associated graded rings are often far from well-behaved. For instance, it has been observed that $G(I)$ and $\overline{G}(I)$ are rarely Cohen–Macaulay even if A is Cohen–Macaulay. A lot of attention in the literature is devoted to obtain criteria for $G(\mathcal{F})$ to be Cohen–Macaulay in terms of Hilbert coefficients and/or reduction number, with particular attention to the filtration $\{I^n\}_{n \in \mathbb{Z}}$ and $\{\overline{I}^n\}_{n \in \mathbb{Z}}$. For instance, it is well-known that if $r(\mathcal{F}) \leq 1$ in a Cohen–Macaulay local ring, then $G(\mathcal{F})$ is Cohen–Macaulay [53]. Also, it is known that if $e_1(\mathcal{F}) = 0$ in a Cohen–Macaulay local ring, then $G(\mathcal{F})$ is Cohen–Macaulay [53]. For these reasons, a lot of research is devoted to obtain a sharp bound on the Hilbert coefficients and reduction number, and studying the behavior of $G(\mathcal{F})$ when these bounds are extremal.

A key ingredient in our approach is the relation between Gorensteinness and the Hilbert series of the associated graded ring. We prove that for Hilbert filtrations over a Gorenstein local ring, once the associated graded ring is Cohen–Macaulay, its Gorensteinness is equivalent to the symmetry of the numerator of its Hilbert series. This reduces the problem to a numerical one and allows us to express Gorensteinness in terms of Hilbert coefficients. We then study

this problem in situations where the reduction number is small, and more generally when a suitable relative reduction number is extremal. In these cases, the Hilbert series can be computed explicitly, which leads to concrete numerical criteria for Gorensteinness. In particular, we obtain characterizations in terms of the first few Hilbert coefficients when the reduction number is at most two, and more generally in cases where the relative reduction number attains its maximal possible value.

Our main results on understanding Gorensteinness of associated graded rings are summarized below: The following result establishes a criterion for Gorensteinness in terms of Hilbert coefficients in a d -dimensional Gorenstein local ring.

Theorem IV. A. *Let $(A, \mathfrak{m})$ be a d -dimensional Gorenstein local ring, $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Suppose $G(\mathcal{F})$ is Cohen-Macaulay. Let*

$$\text{HS}_{G(\mathcal{F})}(z) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d} \quad (h_s \neq 0)$$

be the Hilbert series of $G(\mathcal{F})$. Then $G(\mathcal{F})$ is Gorenstein if and only if $h_{s-i} = h_i$ for all $i = 0, 1, \dots, s$.

We use this result to characterize the Gorensteinness of $G(\mathcal{F})$ in terms of the Hilbert coefficients.

A first result in this direction is obtained when $r(\mathcal{F}) \leq 2$.

Theorem IV. B. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 1$. Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ be a Hilbert filtration.*

- (a) *Let $r(\mathcal{F}) = 1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = 2e_1(\mathcal{F})$.*
- (b) *Suppose there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = e_1(\mathcal{F})$.*

We make an additional assumption when $r(\mathcal{F}) = 2$, namely: there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. With these conditions we prove that $G(\mathcal{F})$ is Cohen-Macaulay, and compute the Hilbert series of $G(\mathcal{F})$. Notice that this condition is not restrictive. For instance, the filtrations $\mathcal{F} = \{\overline{I}^n\}_{n \in \mathbb{Z}}$ and $\mathcal{F} = \{\mathfrak{m}^n\}_{n \in \mathbb{Z}}$ satisfy this criterion. Hence, as a consequence of our result, we obtain a criterion for Gorensteinness of $\overline{G}(I)$. This recovers and extends a recent result by Okuma, Watanabe, and Yoshida in [58].

In order to compute the Hilbert series of $G(\mathcal{F})$, we introduce the notion of *relative reduction number* for Hilbert filtration $\mathcal{F}$. This is inspired by the definition of relative reduction number for normal filtrations defined in [57] in dimension two.

We derive an upper bound for the relative reduction number of a filtration $\text{nr}(\mathcal{F})$ when the depth of $G(\mathcal{F})$ is almost maximal.

We also examine the conditions under which $\text{nr}(\mathcal{F})$ achieves that upper bound, which serves as a crucial technical assumption for the computation of the Hilbert series of $G(\mathcal{F})$. If the upper bound is achieved, then we say the filtration $\mathcal{F}$ has *maximal relative reduction number*. In this case, we are able to compute the Hilbert series of $G(\mathcal{F})$ and obtain a criterion for $G(\mathcal{F})$ to be Gorenstein when $r(\mathcal{F}) \geq 2$. Our main result is summarised below:

Theorem IV. C. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Suppose that $r = \text{nr}(\mathcal{F}) \geq 2$ is maximal. Then*

(a) *The Hilbert series of $G(\mathcal{F})$ is*

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{\lambda + (e_0(\mathcal{F}) - \lambda - 1)t + t^r}{(1 - t)^d}, \text{ where } \lambda = \ell(A/F_1).$$

(b) *$G(\mathcal{F})$ is Gorenstein if and only if $G(\mathcal{F})$ is Cohen-Macaulay and $F_1 = \mathfrak{m}$ (resp. $F_1 = \mathfrak{m}$ and $e_0(\mathfrak{m}) = 2$) if $r = 2$ (resp. $r \geq 3$).*

We include some interesting examples of semigroup rings in dimension one and use our criteria to detect Gorensteinness of $\overline{G}(\mathfrak{m})$.

In the last section of this chapter, we discuss the Gorensteinness of associated graded rings of integral closure filtration of the maximal ideal in a Zariski-type hypersurface ring. Let K be an algebraically closed field, and let $m \geq 1$, $2 \leq a \leq b$ be integers, and put $S = K[[x, y_1, \dots, y_m]]$ be a formal power series ring with $(m + 1)$ -variables over K . Put $\underline{y} = y_1, \dots, y_m$. Let g, f be polynomials such that

$$\begin{aligned} g &= g(\underline{y}) \in (\underline{y})^b \setminus (\underline{y})^{b+1}, \\ f &= f(x, \underline{y}) = x^a - g(\underline{y}). \end{aligned}$$

Then a hypersurface $A = S/(f)$ is called “*Zariski-type hypersurface*” if $a \neq 0$ in K . We give a criterion for $\overline{G}(\mathfrak{m})$ to be Gorenstein in the case where A is a Zariski-type hypersurface.

Theorem IV. D. *$\overline{G}(\mathfrak{m})$ is Gorenstein if and only if $b \equiv 0, d \pmod{a}$ where $d = \gcd(a, b)$. Moreover,*

- (a) *If $b \equiv 0 \pmod{a}$, then $\overline{G}(\mathfrak{m})$ is a reduced, hypersurface.*
- (b) *If $b \equiv d \pmod{a}$ and $d = 1$, then $\overline{G}(\mathfrak{m})$ is a non-reduced, hypersurface.*
- (c) *If $b \equiv d \pmod{a}$ and $d > 1$, then $\overline{G}(\mathfrak{m})$ is a complete intersection but not a hypersurface.*

Chapter 2

Waring decompositions of the product of two quadrics: the small rank cases

In this chapter, we present our study on the Waring decomposition of the product of quadrics. In Sections [2.2](#) and [2.3](#) we study the specific forms $x^2(y_1^2 + \cdots + y_n^2)$ and $(x_1^2 + x_2^2)(y_1^2 + \cdots + y_n^2)$. In particular, we study the structure, the defining ideal, and the Hilbert function of all minimal apolar sets. As a result, we compute the forbidden locus of these forms. In the final Section [2.4](#) of this chapter, we present new lower and upper bounds for the Waring rank of forms of the type $(x_1^2 + \cdots + x_m^2)(y_1^2 + \cdots + y_n^2)$.

2.1 Basic results

In this section, we prove some technical lemmas on upper bounds for Hilbert function and change of coordinates.

For the first result of this section, we set $S = \mathbb{C}[x_0, \dots, x_n]$ be the standard graded polynomial ring and let $T = \mathbb{C}[X_0, \dots, X_n]$ be the ring of differential operators on S . An important tool, which is useful in the study of Waring rank problems is the apolarity action. Let $T = \mathbb{C}[X_0, \dots, X_n]$ be a polynomial ring over $\mathbb{C}$. The apolarity action of T on S is defined on monomials by

$$X_0^{a_0} \cdots X_n^{a_n} \circ (x_0^{b_0} \cdots x_n^{b_n}) = \frac{\partial^{a_0}}{\partial x_0^{a_0}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} (x_0^{b_0} \cdots x_n^{b_n}),$$

and then extended linearly on S and T . For more properties, refer [Appendix C](#).

Given a homogeneous form $F \in S_d$, using the apolarity action, we write annihilator ideal of F as

$$F^\perp = \{g \in T \mid g \circ F = 0\}.$$

Given a Waring decomposition $F = \sum_{i=1}^l L_i^d$ where $L_i = a_{i0}x_0 + a_{i1}x_1 + \cdots + a_{in}x_n$ for all $1 \leq i \leq l$, we associate a set of l distinct points

$$\mathbb{X} = \{[a_{10} : a_{11} : \cdots : a_{1n}], \dots, [a_{l0} : a_{l1} : \cdots : a_{ln}]\} \subset \mathbb{P}^n.$$

The following lemma shows a beautiful connection between the set of points associated to a Waring decomposition and apolarity theory.

Lemma 2.1.1 (Apolarity Lemma, [39]). *Let $F \in S_d$ be a homogeneous polynomial of degree d . Then F has a Waring decomposition $F = \sum_{i=1}^l L_i^d$ if and only if $I(\mathbb{X}) \subset F^\perp$ where $\mathbb{X}$ is the set of distinct points corresponding to the linear forms L_i .*

It is known that $T/F^\perp$ is an Artinian Gorenstein ring. Recall that $\mathbb{X}$ is said to be apolar to F if $I(\mathbb{X}) \subseteq F^\perp$, and we call $I(\mathbb{X})$ an apolar ideal of F . If $\mathbb{X}$ is apolar to F , then by Apolarity Lemma [2.1.1] we obtain $\text{Rk } F \leq |\mathbb{X}|$, where $|\mathbb{X}|$ denotes the cardinality of $\mathbb{X}$. Further, $\mathbb{X}$ is a minimal apolar set of F if $|\mathbb{X}| = \text{Rk } F$.

Obtaining good bounds on the Waring rank is not an easy task. In [12], Carlini et. al. introduced e-computability. This is very useful to obtain lower bound on the Waring rank.

Theorem 2.1.2 (Theorem 3.3 [12]). *Let $F \in S_d$, and $\mathbb{X} \in \mathbb{P}^n$ be an apolar set of F . Let I be an ideal in T generated in degree $e > 0$, and $t \in I_e$. If t is a non-zerodivisor in $T/(I(\mathbb{X}) : I)$, then for $s \gg 0$, we have*

$$|\mathbb{X}| \geq \frac{1}{e} \sum_{i=0}^s \text{HF}(T/(I(\mathbb{X}) : I) + (t), i) \geq \frac{1}{e} \sum_{i=0}^s \text{HF}(T/(F^\perp : I) + (t), i).$$

A homogeneous form $F \in S_d$ is said to be *e-computable* if there exists an ideal $I \subset T$, and $t \in I_e \neq 0$ such that

$$\text{Rk}(F) = \frac{1}{e} \sum_{i=0}^s \text{HF}(T/(F^\perp : I + (t)), i).$$

The study of Waring locus of a form was introduced by Carlini, Catalisano, Oneto in [15]. For $F \in S_d$, Waring locus of F is given by

$$\mathcal{W}_F = \{P \in \mathbb{P}_{\mathbb{C}}^n \mid P \in \mathbb{X}, I(\mathbb{X}) \subset F^\perp, |\mathbb{X}| = \text{Rk}(F)\}.$$

The locus of forbidden points of a form is defined as its complement, F is $\mathcal{F}_F = \mathbb{P}^n \setminus \mathcal{W}_F$.

Remark 2.1.3. If F is e-computable by an ideal I , and $t \in I_e \neq 0$, then Theorem 2.1.2 implies that $V(I) \subseteq \mathcal{F}_F$.

The following lemma is, at the best of our knowledge, a new bound on the Hilbert function of a minimal apolar set to a given form.

Lemma 2.1.4. *Let $\mathbb{X}$ be a minimal set of s points apolar to $F \in S_d$ and let $V \subset (T/I(\mathbb{X}))_a$ be a subspace. If $g \in (T/I(\mathbb{X}))_b$ where $b \geq 0$, is a nonzerodivisor and $gV \subset (F^\perp/I(\mathbb{X}))$, then*

$$\text{HF}(\mathbb{X}, d - a - b) \leq s - \dim_{\mathbb{C}} V.$$

Proof. Let $\mathbb{X}$ be a minimal set of s points apolar to F . For each $P_i = [a_{i,0} : \dots : a_{i,n}] \in \mathbb{X}$ we can construct the linear form

$$L_i = a_{i,0}x_0 + a_{i,1}x_1 + \dots + a_{i,n}x_n \in S$$

such that, by the 2.1.1,

$$F = \sum_{i=1}^s \epsilon_i L_i^d,$$

for some constants $\epsilon_i \in \mathbb{C}$. Note that, since $\mathbb{X}$ is minimal, $\epsilon_i \neq 0$, for all i . Hence we can assume that $\epsilon_i = 1$ for $1 \leq i \leq s$. Let $\dim V = l$, and $v_1, v_2, \dots, v_l$ be a basis of V . Since $gV \subset F^\perp/I(\mathbb{X})$, there exists $f_j \in I(\mathbb{X})$ such that $f_j + gv_j \in F^\perp$. Since $f_j \in I(\mathbb{X}) \subseteq F^\perp$, we get $gv_j \in F^\perp$ for $1 \leq j \leq l$.

Hence for $1 \leq j \leq l$, we have $gv_j \circ F = 0$. Thus, by Proposition A.2 we get the system of equations

$$\sum_{i=1}^s gv_j(P_i) L_i^{d-a-b} = 0$$

for $1 \leq j \leq l$, which in matrix form has the shape:

$$\begin{bmatrix} L_1^{d-a-b} & L_2^{d-a-b} & \dots & L_s^{d-a-b} \end{bmatrix} \begin{bmatrix} gv_1(P_1) & gv_2(P_1) & \dots & gv_l(P_1) \\ gv_1(P_2) & gv_2(P_2) & \dots & gv_l(P_2) \\ \vdots & \vdots & \vdots & \vdots \\ gv_1(P_s) & gv_2(P_s) & \dots & gv_l(P_s) \end{bmatrix} = \mathbf{0}. \quad (2.1.1)$$

If we let $\{u_1, \dots, u_c\}$ be an ordered basis of T_{d-a-b} , where $c = \binom{n+d-a-b}{d-a-b}$, then we can write L_i^{d-a-b} as a linear combination of the chosen basis. Namely, there exists a $c \times s$ matrix

M with complex entries such that

$$\begin{bmatrix} L_1^{d-a-b} & L_2^{d-a-b} & \dots & L_s^{d-a-b} \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \dots & u_c \end{bmatrix} M. \quad (2.1.2)$$

Thus, we have that

$$\text{rank}(M) = \dim_{\mathbb{C}}(\mathbb{C}L_1^{d-a-b} + \mathbb{C}L_2^{d-a-b} + \dots + \mathbb{C}L_s^{d-a-b}). \quad (2.1.3)$$

Substituting (2.1.2) in (2.1.1), we obtain

$$\begin{bmatrix} u_1 & u_2 & \dots & u_c \end{bmatrix} M \begin{bmatrix} gv_j(P_i) \end{bmatrix}_{1 \leq i \leq s, 1 \leq j \leq l} = \mathbf{0}.$$

Since $u_1, \dots, u_c$ is a basis of T_{d-a-b} , we get

$$M \begin{bmatrix} gv_j(P_i) \end{bmatrix}_{1 \leq i \leq s, 1 \leq j \leq l} = \mathbf{0} \quad (2.1.4)$$

Moreover, we get that

$$\begin{bmatrix} gv_j(P_i) \end{bmatrix}_{1 \leq i \leq s, 1 \leq j \leq l} = GN$$

where G is the $s \times s$ diagonal matrix having on the diagonal $g(P_1), \dots, g(P_s)$ and

$$N = \begin{bmatrix} v_j(P_i) \end{bmatrix}_{1 \leq i \leq s, 1 \leq j \leq l}.$$

Note that, since g is a non-zerodivisor in $(T/I(\mathbb{X}))_b$, we get $g(P_i) \neq 0$ for all $1 \leq i \leq s$. Indeed, by relabelling, choose m such that $g(P_i) = 0$ for $1 \leq i \leq m$, $g(P_i) \neq 0$ for $m+1 \leq i \leq s$. Since $\mathcal{P}_i \not\subseteq \mathcal{P}_1$ for all $i = m+1, \dots, |\mathbb{X}|$, we can choose $h \in \mathcal{P}_{m+1} \cap \dots \cap \mathcal{P}_{|\mathbb{X}|} \setminus (\mathcal{P}_1)$, where $\mathcal{P}_i$ denotes the ideal of the point P_i . Note that $I(\mathbb{X}) = \mathcal{P}_1 \cap \dots \cap \mathcal{P}_{|\mathbb{X}|}$. Further, $0 \neq h \in T/I(\mathbb{X})$, and $0 = gh \in T/I(\mathbb{X})$. Since g is a nonzerodivisor in $(T/I(\mathbb{X}))_b$, this is a contradiction. Hence G is invertible. Consider the evaluation map

$$\mathfrak{J} : \mathbb{C}v_1 + \dots + \mathbb{C}v_l \rightarrow \mathbb{C}^s$$

given by

$$f \rightarrow (f(P_1), \dots, f(P_s)).$$

Since $\{v_1, v_2, \dots, v_l\}$ are linearly independent in $T/I(\mathbb{X})$, we obtain $\mathfrak{J}$ is an injective map. Therefore we have

$$\text{rank } \mathfrak{J} = \text{rank}(N) = \dim_k(V) = l. \quad (2.1.5)$$

Thus, Equation (2.1.4) yields

$$MGN = \mathbf{0}.$$

Therefore, the column space of GN is contained in the null space of M . Thus,

$$\text{rank}(GN) \leq \text{nullity}(M),$$

where $\text{nullity}(M)$ is the dimension of the kernel of M . Since G is invertible, GN and N have the same column space. Thus

$$\text{rank}(N) = \text{rank}(GN) \leq \text{nullity}(M).$$

Thus, by the Rank-Nullity Theorem, we have

$$\text{rank}(M) = s - \text{nullity}(M)$$

and thus, $\text{rank}(M) \leq s - \text{rank}(N)$. By Equation (2.1.3), we have

$$\text{rank}(M) = \dim_{\mathbb{C}}(L_1^{d-a-b}, \dots, L_s^{d-a-b}) = \text{HF}(\mathbb{X}, d-a-b)$$

where the last equality follows by [39] Lemma (ii) and the fact that the apolarity action gives a perfect bilinear pairing from $T_{d-a-b} \times S_{d-a-b} \rightarrow \mathbb{C}$. By Equation (2.1.5), we have $\text{rank}(N) = \dim_{\mathbb{C}} V$.

Hence the result is now proved. □

From now on, and throughout the chapter, we consider a specialised version of the rings S and T in which we divide the variables into two sets. Namely, we let $S = \mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]$ be a standard graded polynomial ring and let $T = \mathbb{C}[X_1, \dots, X_m, Y_1, \dots, Y_n]$ the ring of differential operators on S . Let $\mathfrak{m} = (X_1, \dots, X_m, Y_1, \dots, Y_n)$ be the irrelevant maximal ideal in T . An ideal $I \subseteq T$ is said to be saturated if $I = I_{\text{sat}}$ where $I_{\text{sat}} = \bigcup_{i \geq 1} (I : \mathfrak{m}^i)$. Note that saturated ideals in T are in one-to-one correspondence with the closed subschemes of $\mathbb{P}^{m+n-1}$ (see [28] Exercise II.5.10). Throughout this chapter, our interest is a finite set of distinct points in projective space. The defining ideal of such points is a radical ideal, which is saturated. The following lemma shows that after the projection, we have a nice description of the defining ideal of the points obtained from the projection.

Lemma 2.1.5. *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^{m+n-1} = \mathbb{P}(S_1)$. If we let $\mathbb{Y}$ be the projection of $\mathbb{X}$ from $V(Y_1, \dots, Y_n)$ on $V(X_1, \dots, X_m)$, then*

$$I(\mathbb{Y}) = (X_1, \dots, X_m) + I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n].$$

Proof. Note that $I(\mathbb{X})$ is a radical ideal being an ideal of points, and so is $J = I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n]$ as the contraction of a radical ideal is radical. Now we show that $(X_1, \dots, X_m) + J$ is a radical ideal. Observe that

$$\frac{\mathbb{C}[X_1, \dots, X_m, Y_1, \dots, Y_n]}{(X_1, \dots, X_m) + J} \cong \frac{\mathbb{C}[Y_1, \dots, Y_n]}{J}. \quad (2.1.6)$$

Since J is a radical ideal, $\mathbb{C}[Y_1, \dots, Y_n]/J$ is reduced. Hence, by Equation (2.1.6), we have $(X_1, \dots, X_m) + J$ is a radical ideal. By the standard definition of projection, we have that $I(\mathbb{Y})$ is the saturation of $(X_1, \dots, X_m) + J$. Since a radical ideal is saturated, the result is proved. $\square$

The next lemma shows an interesting connection between apolar set of a polynomial and its projection.

Lemma 2.1.6. *Let $F = F_1 F_2 \neq 0$ be such that $F_1 \in S_1 = \mathbb{C}[x_1, \dots, x_m]$ and $F_2 \in S_2 = \mathbb{C}[y_1, \dots, y_n]$, and $\mathbb{X}$ be apolar to F . If $\mathbb{Y}_1$ is a projection of $\mathbb{X}$ from $V(X_1, \dots, X_m)$ to $V(Y_1, \dots, Y_n)$, then $\mathbb{Y}_1$ is apolar to F_1 . Similarly, If $\mathbb{Y}_2$ is a projection of $\mathbb{X}$ from $V(Y_1, \dots, Y_n)$ to $V(X_1, \dots, X_m)$, then $\mathbb{Y}_2$ is apolar to F_2 .*

Proof. By [22, Proposition 3.1] we have

$$F^\perp = (F_1^\perp)_{S_1} + (F_2^\perp)_{S_2}$$

where $(F_i^\perp)_{S_i}$ denotes the annihilator of F_i computed in S_i for $i = 1, 2$. Let $\mathbb{Y}_1$ be a projection of $\mathbb{X}$ from $V(X_1, \dots, X_m)$ to $V(Y_1, \dots, Y_n)$. By Lemma 2.1.5 we have

$$I(\mathbb{Y}_1) = I(\mathbb{X}) \cap \mathbb{C}[X_1, \dots, X_m] + (Y_1, \dots, Y_n).$$

If $g \in I(\mathbb{X}) \cap \mathbb{C}[X_1, \dots, X_m]$, then $g \circ F = 0$. This implies $F_2(g \circ F_1) = 0$. Since $F_2 \neq 0$, we obtain $g \in F_1^\perp$. Moreover, $(Y_1, \dots, Y_n) \subseteq F_1^\perp$. Hence $I(\mathbb{Y}_1) \subset F_1^\perp$. Therefore $\mathbb{Y}_1$ is apolar to F_1 . Similarly, we obtain $\mathbb{Y}_2$ is apolar to F_2 . $\square$

To study minimal apolar sets to a form F , we use the Apolarity Lemma. Thus, we need to describe the annihilators of the forms of our interest. This is done in the following lemma.

Lemma 2.1.7. *Let $F = (x_1^2 + \dots + x_m^2)(y_1^2 + \dots + y_n^2)$, where $m \geq 1, n \geq 2$. Then*

$$F^\perp = \begin{cases} (X_1^3, \{Y_1^2 - Y_i^2\}_{2 \leq i \leq n}, \{Y_i Y_j\}_{1 \leq i < j \leq n}) & \text{if } m = 1 \\ (\{X_1^2 - X_i^2\}_{2 \leq i \leq m}, \{X_i X_j\}_{1 \leq i < j \leq m}, \{Y_1^2 - Y_i^2\}_{2 \leq i \leq n}, \{Y_i Y_j\}_{1 \leq i < j \leq n}) & \text{if } m \geq 2. \end{cases}$$

Proof. Let

$$I = \begin{cases} (X_1^3, \{Y_1^2 - Y_i^2\}_{2 \leq i \leq n}, \{Y_i Y_j\}_{1 \leq i < j \leq n}) & \text{if } m = 1 \\ (\{X_1^2 - X_i^2\}_{2 \leq i \leq m}, \{X_i X_j\}_{1 \leq i < j \leq m}, \{Y_1^2 - Y_i^2\}_{2 \leq i \leq n}, \{Y_i Y_j\}_{1 \leq i < j \leq n}) & \text{if } m \geq 2 \end{cases}$$

and $T = \mathbb{C}[X_1, \dots, X_m, Y_1, \dots, Y_n]$. Note that $I \subseteq F^\perp$. Moreover, a standard computation yields:

i	0	1	2	3	4
$\text{HF}(T/I, i)$	1	$n + m$	$nm + 2$	$n + m$	1

Note that $T/F^\perp$ is an Artinian Gorenstein ring of socle degree 4. Thus, by symmetry, we have that

$$\text{HF}(T/F^\perp, 0) = \text{HF}(T/F^\perp, 4) = 1 \text{ and } \text{HF}(T/F^\perp, 1) = \text{HF}(T/F^\perp, 3) = n + m.$$

Now we compute $\text{HF}(T/F^\perp, 2)$ by acting on all possible degree 2 forms on F . This can be done since apolarity gives a perfect pairing (see [26]) between S_2 and T_2 . For $1 \leq i \leq m$ and $1 \leq j \leq n$, we have

$$\begin{aligned} X_i Y_j \circ F &= 4x_i y_j \\ X_i^2 \circ F &= y_1^2 + \dots + y_n^2 \\ Y_j^2 \circ F &= x_1^2 + \dots + x_m^2. \end{aligned}$$

Thus, $\text{HF}(T/F^\perp, 2) = mn + 2$, and hence T/I and $T/F^\perp$ have the same Hilbert function. Since $I \subseteq F^\perp$, this implies $I = F^\perp$. □

We will often use group actions to study apolar subsets. In particular, given the ring

$$\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]$$

we will consider the following action. We denote by $O(n)$ the subgroup of the complex orthogonal group $O(\mathbb{C}, m + n)$ acting as the identity on the m variables x_i and in the standard way on the n variables y_j . To a given linear form

$$L = a_1 x_1 + \dots + a_m x_m + b_1 y_1 + \dots + b_n y_n$$

corresponds to a projective point

$$L_P = [a_1 : \dots : a_m : b_1 : \dots : b_n],$$

and the action of $g \in O(n)$ on L corresponds to the action of g^T on L_P . Indeed,

$$g(L) = [a_1, \dots, a_m, b_1, \dots, b_n] \cdot g \cdot [\mathbf{x}, \mathbf{y}]^T = (g^T \cdot [a_1, \dots, a_m, b_1, \dots, b_n]^T)^T \cdot [\mathbf{x}, \mathbf{y}]^T.$$

Note that the coefficient of linear form L after the action of g becomes $g^T L_P^T$. Hence, if $\mathbb{X}$ is apolar to a form F and $g \in O(n)$, then $g^T(\mathbb{X})$ is apolar to $g(F)$.

In particular, we will use the following technical lemma to avoid zeroes in apolar subsets.

Lemma 2.1.8. *Let $F \in \mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]$ be a form fixed by the action of $O(n)$, and let $\mathbb{X}$ be a set of r points apolar to F such that $\mathbb{X} \cap V(Y_1, \dots, Y_n) = \emptyset$. Then, there exists a set $\mathbb{Y}$ of r points apolar to F such that, if*

$$[a_1 : \dots : a_m : b_1 : \dots : b_n] \in \mathbb{Y},$$

then $b_j \neq 0$ for all j .

Proof. For $P = [a'_1 : \dots : a'_m : c_1 : \dots : c_n] \in \mathbb{X}$ and $g \in O(n)$ we set

$$g^T(P) = [a_1 : \dots : a_m : b_1 : \dots : b_n]$$

and we define

$$U(P, j) = \{g \in O(n) \mid b_j = 0\}.$$

Note that $U(P, j)$ is defined by a linear equation in $O(n)$. Since $P \notin V(Y_1, \dots, Y_n)$ there exists $h \in O(n)$ such that $h^T c_j \neq 0$. Hence, $U(P, j)$ is a proper subvariety of $O(n)$. If we let

$$U = \bigcup_{P \in \mathbb{X}, 1 \leq j \leq n} U(P, j),$$

then U is a proper subvariety of $O(n)$. Let $g \in O(n) \setminus U$ and let

$$\mathbb{Y} = g^T(\mathbb{X}).$$

Then by construction $\mathbb{Y}$ is a set of r points with the required property, and the proof is now completed. $\square$

We will often deal with a finite set of points in general linear position, that is, each of its subsets has the maximal possible linear span. Thus we recall the following basic fact:

Remark 2.1.9. If $\mathbb{Y} = \{P_1, \dots, P_n\}$ be a set of n points in general position in $\mathbb{P}^{n-1}$, then there exist linearly independent linear forms $L_1, \dots, L_n$ such that

$$I(\mathbb{Y}) = (L_i L_j \mid 1 \leq i < j \leq n).$$

Proof. First, we prove the existence of L_i . Let $P_i = [a_{i1} : \dots : a_{in}]$, and $v_i = (a_{i1}, \dots, a_{in})^T$ be the column vector corresponding to P_i . Let $A_i = (v_1 \dots \hat{v}_i \dots v_n)^T$ be the matrix of size $(n-1) \times n$ corresponding to the points $P_1, \dots, \hat{P}_i, \dots, P_n$. Since the points $P_1, \dots, P_n$ are in general position, by definition A has rank $n-1$. Therefore, the homogeneous system

$$AX = 0$$

has a nontrivial solution, say $(\alpha_{i1}, \dots, \alpha_{in})$. Then define $L_i = \alpha_{i1}Y_1 + \dots + \alpha_{in}Y_n$. Hence $L_j \in I(P_i)$ for all $j \neq i$. Note that $L_i(P_i) \neq 0$ since $P_1, \dots, P_n$ are in general position. Moreover, $\text{ht}(L_1, \dots, \hat{L}_i, \dots, L_n) = \text{ht}I(P_i)$ and both are prime ideals. Hence $(L_1, \dots, \hat{L}_i, \dots, L_n) = I(P_i)$ for all $1 \leq i \leq n$. Now we show that $\{L_1, \dots, L_n\}$ is linearly independent. Suppose $a_1L_1 + \dots + a_nL_n = 0$. Then for all $i = 1, \dots, n$,

$$a_i = \frac{1}{L_i(P_i)}(a_1L_1 + \dots + a_nL_n)(P_i) = 0.$$

Hence $L_1, \dots, L_n$ are linearly independent. Moreover,

$$\begin{aligned} I(\mathbb{Y}) &= \bigcap_{i=1}^n I(P_i) \\ &= \bigcap_{i=1}^n (L_1, \dots, \hat{L}_i, \dots, L_n) \\ &= (L_iL_j \mid 1 \leq i < j \leq n). \end{aligned}$$

□

We conclude this section with a technical result that we will often use to make computations easier. Note that the following lemma implies the well-known structure of the orbits of action of the complex orthogonal group.

Lemma 2.1.10. *Let $G = y_1^2 + \dots + y_n^2$ and let $\mathbb{E} = \{Q_1, \dots, Q_n\} \subset \mathbb{P}^{n-1}$ be a minimal apolar set of G . Let O be the $n \times n$ matrix whose i^{th} column is given by a choice of coordinate of Q_i . There exists a choice of coordinates for each Q_i such that O is a complex orthogonal matrix.*

Proof. Let $Q_i = [a_{i1} : \dots : a_{in}]$, $1 \leq i \leq n$. Since we are working over $\mathbb{C}$, any minimal sum of power decomposition of G has the shape

$$G = \sum_{i=1}^n L_i^2 \tag{2.1.7}$$

where $L_i = a_{i1}y_1 + \dots + a_{in}y_n$. Note that the Apolarity Lemma also states that the converse is true, that is, given the set of apolar points Q_i , after rescaling, we can build the forms L_i in

Equation (2.1.7). Since any quadratic form is completely determined by its associated symmetric matrix, we can write (2.1.7) in matrix form

$$I_n = \sum_{i=1}^n [a_{ij}a_{il}]_{1 \leq j, l \leq n} \quad (2.1.8)$$

where I_n is the $n \times n$ identity matrix and $[a_{ij}a_{il}]_{1 \leq j, l \leq n}$ is the matrix associated to L_i^2 . If we let $O = (a_{ji})_{1 \leq i, j \leq n}$, then (2.1.8) yields

$$OO^\top = I_n.$$

Hence, the result is proved. □

2.2 The form $x^2(y_1^2 + \cdots + y_n^2)$

In this section, we focus on the case $F = x^2(y_1^2 + \cdots + y_n^2)$ with $n \geq 2$. Thus, we consider the ring $S = \mathbb{C}[x, y_1, \dots, y_n]$ and $T = \mathbb{C}[X, Y_1, \dots, Y_n]$ the ring of differential operator on S . We recall that, by [12, Proposition 4.4], we know that $\text{Rk } F = 3n$.

We will now proceed to investigate the structure of any minimal apolar sets of F .

Proposition 2.2.1. *Let $F = x^2(y_1^2 + \cdots + y_n^2)$ with $n \geq 2$ and let $\mathbb{X}$ be a set of $\text{Rk } F = 3n$ distinct points apolar to F . Then the following holds:*

- i) *If $\mathbb{Y}$ is the projection of $\mathbb{X}$ from $V(Y_1, \dots, Y_n)$ on $V(X)$, then $\mathbb{Y}$ is a set of n distinct points in linear general position, that is*

$$\begin{array}{c|c|c|c} i & 0 & 1 & 2 \\ \hline \text{HF}(\mathbb{Y}, i) & 1 & n & n \end{array},$$

and $\text{HF}(\mathbb{Y}, i) = n$ for $i \geq 3$.

- ii) *Let $\mathbb{Y} = \{Q_1, \dots, Q_n\}$ and $P = [1 : 0 : \dots : 0]$. If $\Lambda_i = \langle Q_i, P \rangle$, where $\langle Q_i, P \rangle$ denotes the linear span of Q_i and P , then $\mathbb{X} \cap \Lambda_i$ is a set of three distinct points for $1 \leq i \leq n$.*

Proof. We start by proving (i). By Lemma 2.1.7 we have

$$F^\perp = (X^3, \{Y_i Y_j\}_{(1 \leq i < j \leq n)}, \{Y_1^2 - Y_i^2\}_{(2 \leq i \leq n)}). \quad (2.2.1)$$

Let $I(\mathbb{X})$ be an ideal of a minimal apolar set $\mathbb{X}$. Since $O(n)$ fixes F , by Lemma 2.1.8 we may assume that for $P_i \in \mathbb{X}$, $P_i = [b_i : a_{i,1} : \dots : a_{i,n}]$ is such that $a_{i,1} \neq 0$ for $1 \leq i \leq n$. By Lemma

A.3 we obtain Y_1 is a non-zerodivisor in $T/I(\mathbb{X})$. Let $V := \langle Y_2, Y_3, \dots, Y_n \rangle_{\mathbb{C}} \subset (T/I(\mathbb{X}))_1$. Since $I(\mathbb{X}) \subset F^\perp$, we have that $I(\mathbb{X})_1 = 0$. Thus $\dim V = n - 1$, and $Y_1 V \subset (F^\perp/I(\mathbb{X}))$. Hence, by Lemma **2.1.4**, we obtain that

$$\text{HF}(\mathbb{X}, 2) \leq 3n - (n - 1) = 2n + 1, \quad (2.2.2)$$

and that

$$\dim I(\mathbb{X})_2 \geq \binom{n+2}{2} - (2n + 1) = \binom{n}{2}. \quad (2.2.3)$$

Note that, by Equation **(2.2.1)**, $F^\perp$ does not involve the variable X in any degree 2 terms. Thus we get

$$I(\mathbb{X})_2 = (I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n])_2. \quad (2.2.4)$$

Therefore by Equation **(2.2.3)** we obtain

$$\dim(I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n])_2 \geq \binom{n}{2}. \quad (2.2.5)$$

Let $\mathbb{Y}$ be the projection of $\mathbb{X}$ from the point $P = [1 : 0 : \dots : 0]$ on the hyperplane $V(X)$. By Lemma **2.1.5** we have

$$I(\mathbb{Y}) = (X) + I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n].$$

Since $I(\mathbb{X})_1 = 0$, we get $I(\mathbb{Y})_1 = (X)_1$ and thus

$$\text{HF}(\mathbb{Y}, 1) = n.$$

Since $I(\mathbb{Y})_2 = (X)_2 + (I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n])_2$, Equation **2.2.5** yields

$$\text{HF}(\mathbb{Y}, 2) \leq \binom{n+2}{2} - \left((n+1) + \binom{n}{2} \right) = n. \quad (2.2.6)$$

Since $\text{HF}(\mathbb{Y}, 1) = n$ and $I(\mathbb{Y})$ is an ideal of reduced points we obtain $\text{HF}(\mathbb{Y}, 2) = n$. Hence we get the Hilbert function of $\mathbb{Y}$ as

$$\begin{array}{c|c|c|c|c} i & 0 & 1 & 2 & \dots \\ \hline \text{HF}(\mathbb{Y}, i) & 1 & n & n & n \end{array} \quad (2.2.7)$$

Therefore $\mathbb{Y}$ is a set of n points in a general position. This concludes the proof of (i).

We now prove (ii). Let $\mathbb{Y} = \{Q_1, \dots, Q_n\}$. We have

$$\mathbb{X} \subset \bigcup_{i=1}^n \langle Q_i, P \rangle.$$

Let $\mathcal{H}_i = \langle P, Q_1, \dots, \hat{Q}_i, \dots, Q_n \rangle$ be the hyperplane generated by the linear span of P and the points $Q_1, \dots, \hat{Q}_i, \dots, Q_n$. Note that, since $(F^\perp)_1 = 0$, $Q_i \notin \mathcal{H}_i$. Let $I(\mathcal{H}_i) = L_i$. If we let $L_i = \sum_{j=1}^n \alpha_{ij} Y_j$ for some $\alpha_{ij} \in \mathbb{C}$, then

$$\begin{aligned} F^\perp : I(\mathcal{H}_i) = F^\perp : L_i &= ((\alpha_{i1} Y_1 + \dots + \alpha_{in} Y_n) \circ F)^\perp \\ &= (2x^2(\alpha_{i1} y_1 + \dots + \alpha_{in} y_n))^\perp, \end{aligned}$$

where the second equality holds by [12, Lemma 3.2]. Note that

$$I(\mathbb{X} \cap \Lambda_i) \subseteq I(\mathbb{X}) : I(\mathcal{H}_i) \subseteq F^\perp : I(\mathcal{H}_i) = G^\perp,$$

where $G = 2x^2(\alpha_{i1} y_1 + \dots + \alpha_{in} y_n)$. After a change of variables, G is a monomial and, by [13, Proposition 3.1], we have that $\text{Rk } G = 3$. Therefore each of the n lines Λ_i contains at least 3 points of $\mathbb{X}$. Note that P is the only point in the intersection of any two lines. Moreover, F is e-computable by $I = (Y_1, \dots, Y_n)$ and $t = Y_1 + \dots + Y_n$ (see [12, Proposition 4.4]). Hence by Remark 2.1.3 we obtain $P \in \mathcal{F}_F$. Thus we have at least $3n$ points in the union of n lines. Hence, since $|\mathbb{X}| = 3n$, each line contains exactly 3 points. This concludes the proof of (ii).

The proof of is now completed. □

We can now characterize the points that appear in a minimal apolar set to F . Recall that for $F \in S_d$, the Waring locus of F is defined as

$$\mathcal{W}_F = \{P \in \mathbb{P}_{\mathbb{C}}^n \mid P \in \mathbb{X}, I(\mathbb{X}) \subseteq F^\perp, |\mathbb{X}| = \text{Rk } F\}.$$

The locus of forbidden points of a form is defined as its complement, that is $\mathcal{F}_F = \mathbb{P}^n \setminus \mathcal{W}_F$.

Theorem 2.2.2. *If $F = x^2(y_1^2 + \dots + y_n^2)$, then the forbidden locus of F is given by*

$$\mathcal{F}_F = V(Y_1^2 + \dots + Y_n^2).$$

Proof. Let $Q \in \mathcal{W}_F$ be a point of the Waring locus of F , that is there exists a minimal apolar set $\mathbb{X}$ to F , such that $Q \in \mathbb{X}$. Let $\mathbb{Y}$ be the projection of $\mathbb{X}$ from $V(Y_1, \dots, Y_n) = \{P = [1 : 0 : \dots : 0]\}$ on $V(X)$. By Proposition 2.2.1 we know that $\mathbb{Y}$ is a set n -points in general position. Thus, by Lemma 2.1.7 $\mathbb{Y}$ is a minimal apolar set of $G = y_1^2 + \dots + y_n^2$. By [15, Proposition 3.1] we have $\mathcal{F}_G = V(Y_1^2 + \dots + Y_n^2)$, thus $Q \notin V(Y_1^2 + \dots + Y_n^2)$. Hence we proved that

$$\mathcal{F}_F \supseteq V(Y_1^2 + \dots + Y_n^2).$$

We will now show that equality holds, that is, we prove that if $Q \notin V(Y_1^2 + \cdots + Y_n^2)$, then $Q \in \mathcal{W}_F$. We split the proof in two cases depending on the first coordinate of Q to be zero or not.

- **Case (i):** Let $Q = [0 : \alpha_1 : \cdots : \alpha_n] \notin V(Y_1^2 + \cdots + Y_n^2)$. Hence $\sum_{i=1}^n \alpha_i^2 \neq 0$. Consider $Q' = [\alpha_1 : \cdots : \alpha_n] \in \mathbb{P}^{n-1}$. Since $\sum_{i=1}^n \alpha_i^2 \neq 0$ by [15, Proposition 3.1] we have $Q' \in \mathcal{W}_G$. Thus, there exists a minimal apolar set $\mathbb{D} := \{Q'_1 = Q', Q'_2, \dots, Q'_n\}$ of $G = y_1^2 + \cdots + y_n^2$ containing Q' . By Lemma 2.1.10 we obtain the orthogonal matrix N . Consider

$$I(\mathbb{X}) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X^3 + X(Y_1^2 + \cdots + Y_{n-1}^2 + (-n+1)Y_n^2))$$

an ideal of $3n$ points apolar to F . Note that $P' = [0 : 1 : 0 : \cdots : 0] \in \mathbb{X}$. Further the action of the matrix $g = \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix} \in O(n+1)$ fixes F . Since the action of A takes P' to Q we get the new apolar set $\mathbb{X}' = g^T(\mathbb{X})$ of F containing Q . Hence $Q \in \mathcal{W}_F$.

- **Case (ii):** Let $Q = [1 : \alpha_1 : \cdots : \alpha_n] \notin V(Y_1^2 + \cdots + Y_n^2)$. In this case, we consider the ideal of $3n$ points apolar to F given by

$$I(\mathbb{X}') = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X^3 + (-\sum_{i=1}^n Y_i)(Y_1^2 + \cdots + Y_{n-1}^2 + (-n+1)Y_n^2)).$$

Moreover, the point $P'' = [1 : 1 : 0 : \cdots : 0] \in \mathbb{X}'$. We construct a matrix A similarly to case (i) where the action of A fixes F and A takes P'' to Q .

The proof is now completed. □

We now focus on algebraic properties of minimal apolar sets to F : we begin by describing their Hilbert functions.

Proposition 2.2.3. *If $F = x^2(y_1^2 + \cdots + y_n^2)$, $n \geq 2$, and $\mathbb{X}$ is a set of $3n$ points apolar to F , then the Hilbert function of $\mathbb{X}$ is*

i	0	1	2	3	4	$\cdots$
$\text{HF}(\mathbb{X}, i)$	1	$n+1$	$2n+1$	$3n$	$3n$	$3n$

Proof. Since $I(\mathbb{X}) \subset F^\perp$ and $(F^\perp)_1 = 0$, we obtain $\text{HF}(\mathbb{X}, 1) = n+1$. Now, we claim that $\text{HF}(\mathbb{X}, 2) = 2n+1$.

Proof of the claim. Note that by Equation [2.2.2](#) we have $\text{HF}(\mathbb{X}, 2) \leq 2n + 1$. By contradiction, we assume that $\text{HF}(\mathbb{X}, 2) \leq 2n$ and we use the notation of the proof of Proposition [2.2.1](#). From the assumption we get $\dim I(\mathbb{X})_2 \geq \binom{n+2}{2} - 2n$. Since $F_2^\perp$ does not involve any forms in the variable X , so is $I(\mathbb{X})_2$. Therefore, $\dim I(\mathbb{X})_2 \cap \mathbb{C}[Y_1, Y_2, \dots, Y_n] \geq \binom{n+2}{2} - 2n = \binom{n}{2} + 1$. This implies $\text{HF}(\mathbb{Y}, 2) \leq n - 1$, a contradiction to Equation [2.2.7](#). $\square$

We now prove by contradiction that $\text{HF}(\mathbb{X}, 3) = 3n$. Since $\mathbb{X}$ is a set of $3n$ points, we have that $\text{HF}(\mathbb{X}, 3) \leq 3n$. Thus, we assume by contradiction that $\text{HF}(\mathbb{X}, 3) \leq 3n - 1$. Let $W = I(\mathbb{X})_3$. Then

$$\dim W = \dim I(\mathbb{X})_3 \geq \binom{n+3}{3} - 3n + 1. \quad (2.2.8)$$

By Proposition [2.2.1](#)(ii) we know that $\mathbb{X}$ is supported on $\mathcal{L}$ where $\mathcal{L}$ is the union of n lines passing through the point $P = [1 : 0 : \dots : 0]$ and each line in $\mathcal{L}$ contains exactly 3 points of $\mathbb{X}$. Let $W' \subseteq W$ such that every element of W' passes through the point P . Hence

$$\dim W' \geq \dim W - 1. \quad (2.2.9)$$

Note that $W' \subset T_3$ and that each of its elements vanishes on 4 points of each line of $\mathcal{L}$. Thus W' vanishes on $\mathcal{L}$. Let $W'' \subset W'$ be the set of all elements passing through $\binom{n+2}{3} - n$ general points of the hyperplane $V(X)$. Note that

$$\dim W'' \geq \dim W' - \left(\binom{n+2}{3} - n \right) \geq \binom{n}{2} + 1 \quad (2.2.10)$$

where the last inequality follows from Equations [\(2.2.8\)](#) and [\(2.2.9\)](#). Since $\mathcal{L} \cap V(X)$ consists of n points, each element of W'' has X as a factor. Thus, we obtain

$$W'' = \langle XQ_1, \dots, XQ_j \rangle$$

where $j \geq \binom{n}{2} + 1$ and the forms Q_i are linearly independent degree two forms. Since $V(X) \not\supset \mathcal{L}$, we have

$$V(Q_1, Q_2, \dots, Q_j) \supset \mathcal{L}$$

for some $j \geq \binom{n}{2} + 1$. Thus, $Q_1, \dots, Q_j \in I(\mathcal{L})$. Note that $\mathcal{L}$ is the non-degenerate union of n lines through P , thus

$$I(\mathcal{L}) = (L_i L_j : 1 \leq i < j \leq n),$$

where the linear forms L_i do not involve X . In particular, the ideal of $\mathcal{L}$ is generated by $\binom{n}{2}$ degree two forms. Thus a contradiction, since the forms Q_i are linearly independent. Hence $\text{HF}(\mathbb{X}, 3) = 3n$. Since $\mathbb{X}$ is a set of $3n$ reduced points, the proof is completed. $\square$

We now complete the algebraic description of minimal apolar sets to F by describing their defining ideals.

Proposition 2.2.4. *Let $F = x^2(y_1^2 + \cdots + y_n^2)$ with $n \geq 2$. If $\mathbb{X}$ is a minimal apolar set of F , then there exists $\sigma \in O(n)$ such that the ideal of $I(\sigma(\mathbb{X}))$ is given by*

$$I(\sigma(\mathbb{X})) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X^3 + \sum_{i=1}^n L_i(Y_1^2 - Y_i^2))$$

for some linear form L_i in $\mathbb{C}[X, Y_1, \dots, Y_n]$ for $1 \leq i \leq n$.

Proof. By Proposition 2.2.3, we have $\text{HF}(\mathbb{X}, 3) = 3n$. Keeping the same notation as in Proposition 2.2.3, we have $\text{HF}(\mathbb{X}, 2) = 2n + 1$ and $\mathbb{Y}$ is an ideal of n points in general position apolar to $G = y_1^2 + \cdots + y_n^2$. Hence by Lemma 2.1.10, we can assume $\mathbb{Y} = \{e_1, e_2, \dots, e_n\}$. Hence

$$I(\mathbb{Y}) = (Y_i Y_j \mid 1 \leq i < j \leq n) \subseteq I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n].$$

Therefore

$$I(\mathbb{X})_2 = (Y_i Y_j \mid 1 \leq i < j \leq n).$$

Further, by Proposition 2.2.1 we have $\mathbb{X} \subset \mathcal{L}$. Hence $I(\mathbb{X}) \supset I(\mathcal{L})$. Note that $\text{HF}(\mathcal{L}, 3) = \binom{n+3}{3} - (3n+1)$. Hence, there exists exactly one degree 3 generator in $I(\mathbb{X})$. Since $I(\mathbb{X}) \subset F^\perp$, a degree 3 generator has to be of the form $X^3 + \sum_{i=2}^n L'_i(Y_1^2 - Y_i^2)$ for some $L'_i \in T_1$, $2 \leq i \leq n$. We write

$$I(\mathbb{X}) = \bigcap_{i=1}^n (Y_1, Y_2, \dots, \hat{Y}_i, \dots, Y_n, Q = X^3 + \sum_{i=2}^n L'_i(Y_1^2 - Y_i^2)) + J,$$

where J is generated in degree at least four in $I(\mathbb{X})$. By Proposition 2.2.1, we have 3 points on each line $V(Y_1, \dots, \hat{Y}_i, \dots, Y_n)$ for all $1 \leq i \leq n$. Consider $\bar{Q} \in \frac{\mathbb{C}[X, Y_1, \dots, Y_n]}{(Y_1, \dots, \hat{Y}_i, \dots, Y_n)}$. Note that $\bar{Q} = X^3 + (\alpha X + k Y_i) Y_i^2$ with $(\alpha, k) \neq 0$, and $\bar{Q}$ does not have a multiple factor. Indeed, if $\bar{Q}$ has a multiple factor, then there exists a degree 2 form in $(\bar{Q}) + \bar{J}$. Since $(\bar{Q}) + \bar{J}$ is an ideal of 3 points, this is a contradiction. Hence $\bar{Q}$ does not have a multiple factor. Therefore, $(\bar{Q})$ is an ideal of 3 points. Thus $V(\bar{Q}) = V((\bar{Q}) + \bar{J})$. Therefore $(\bar{Q}) \supseteq \bar{J}$. Hence $I(\mathbb{X})$ has generators in degree at most 3. $\square$

Theorem I. A is now proved by Propositions 2.2.1, 2.2.3, 2.2.4 and Theorem 2.2.2.

2.3 The form $(x_1^2 + x_2^2)(y_1^2 + \cdots + y_n^2)$

We note that up to a change of coordinates, the forms $(x_1^2 + x_2^2)(y_1^2 + \cdots + y_n^2)$ and $x_1x_2(y_1^2 + \cdots + y_n^2)$ can be identified. Hence, throughout this section, we assume the form to be $F = x_1x_2(y_1^2 + \cdots + y_n^2)$ with $n \geq 2$. By Proposition 2.4.4, we obtain $\text{Rk } F = 4n$. We study the structure of minimal apolar sets of F , we obtain the forbidden locus of F , and we also describe their ideals and Hilbert functions.

We begin by studying the projection of any minimal apolar set from $V(Y_1, \dots, Y_n)$. A summary of the below result can be visualized in Figure 1.2.

Proposition 2.3.1. *Let $F = x_1x_2(y_1^2 + \cdots + y_n^2)$ and let $\mathbb{X}$ be a minimal apolar set of points to F . Then $\text{Rk } F = 4n$ and the following facts hold:*

- i) *If $\mathbb{Y}$ is the projection of $\mathbb{X}$ from $V(Y_1, \dots, Y_n)$ on $V(X_1, X_2)$, then $\mathbb{Y}$ is a set of n distinct points in linear general position, that is*

$$\begin{array}{c|c|c|c} i & 0 & 1 & 2 \\ \hline \text{HF}(\mathbb{Y}, i) & 1 & n & n \end{array},$$

and $\text{HF}(\mathbb{Y}, i) = n$ for $i \geq 3$.

- ii) *Let $\mathbb{Y} = \{Q_1, \dots, Q_n\}$ and let $\ell = V(Y_1, \dots, Y_n)$. If we consider the plane $\Lambda_i = \langle \ell, Q_i \rangle$, where $\langle \ell, Q_i \rangle$ denotes the linear span of ℓ and Q_i , then $\mathbb{X} \cap \Lambda_i$ is a set of four distinct points for $1 \leq i \leq n$ contained in two lines passing through Q_i .*

Proof. Proposition 2.4.4 immediately yields that $\text{Rk } F = 4n$. We now prove (i). By Lemma 2.1.7 we have

$$F^\perp = (X_1^2, X_2^2, \{Y_i Y_j\}_{(1 \leq i < j \leq n)}, \{Y_1^2 - Y_i^2\}_{(2 \leq i \leq n)}). \quad (2.3.1)$$

Let $\mathbb{X}$ be a minimal apolar set of F . Since $O(n)$ fixes F by Lemma 2.1.8, we can assume that for $P_i = [u_i : v_i : a_{i,1} : \dots : a_{i,n}] \in \mathbb{X}$, $a_{i,1} \neq 0$. Therefore by Lemma A.3 we obtain $g = Y_1 \in (T/I(\mathbb{X}))_1$ is a non-zerodivisor in $T/I(\mathbb{X})$. Let $V = \langle Y_2, Y_3, \dots, Y_n \rangle_{\mathbb{C}} \subset (T/I(\mathbb{X}))_1$. Note that $gV \subset (F^\perp/I(\mathbb{X}))$, and $\dim V = n - 1$. By Lemma 2.1.4, we get

$$\text{HF}(\mathbb{X}, 2) \leq 4n - (n - 1) = 3n + 1.$$

Thus,

$$\dim I(\mathbb{X})_2 \geq \binom{n+3}{2} - 3n - 1. \quad (2.3.2)$$

Let $\mathbb{Y}$ be the projection of $\mathbb{X}$ on $V(X_1, X_2)$ from the line $V(Y_1, \dots, Y_n)$. Therefore by Lemma 2.1.5 we have

$$I(\mathbb{Y}) = (X_1, X_2) + (I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n]). \quad (2.3.3)$$

Since $I(\mathbb{X}) \subset F^\perp$, $I(\mathbb{X})_1 = 0$. Hence $I(\mathbb{Y})_1 = (X_1, X_2)_1$. Therefore

$$\text{HF}(\mathbb{Y}, 1) = n. \quad (2.3.4)$$

We have $I(\mathbb{Y})_2 = (X_1, X_2)_2 + (I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n])_2$. We now use Grassmann formula to estimate

$$\dim I(\mathbb{X})_2 \cap \mathbb{C}[Y_1, \dots, Y_n].$$

By Equation (2.3.1) we note that $\dim(F^\perp)_2 = \binom{n}{2} + (n-1) + 2$ and

$$\dim(F^\perp)_2 \cap \mathbb{C}[Y_1, \dots, Y_n] = \binom{n}{2} + (n-1).$$

Thus, we get that

$$\dim I(\mathbb{X})_2 \cap \mathbb{C}[Y_1, \dots, Y_n] \geq \dim I(\mathbb{X})_2 + \dim(F^\perp)_2 \cap \mathbb{C}[Y_1, \dots, Y_n] - \dim(F^\perp)_2 \geq \binom{n}{2}, \quad (2.3.5)$$

and also

$$\text{HF}(\mathbb{Y}, 2) \leq \binom{n+1}{2} - \binom{n}{2} = n. \quad (2.3.6)$$

Since $I(\mathbb{Y})$ is an ideal of points and $\text{HF}(\mathbb{Y}, 1) = n$, we obtain

$$\text{HF}(\mathbb{Y}, 2) = n.$$

Hence the Hilbert function of $\mathbb{Y}$ is given by

i	0	1	2	$\dots$
$\text{HF}(\mathbb{Y}, i)$	1	n	n	n

Therefore, $\mathbb{Y}$ is a set of n points in a general position.

We can now proceed with the proof of (ii). Let $\mathbb{Y} = \{Q_1, \dots, Q_n\}$. From part (i), we have the following:

$$\mathbb{X} \subset \bigcup_{i=1}^n \langle \ell, Q_i \rangle,$$

where $\langle \ell, Q_i \rangle$ is the plane that passes through ℓ and Q_i . Let $P_1 = [1 : 0 : \dots : 0]$, $P_2 = [0 : 1 : 0 : \dots : 0]$, and $\mathcal{H}_i = \langle P_1, P_2, Q_1, \dots, \hat{Q}_i, \dots, Q_n \rangle$ for $1 \leq i \leq n$. Since $\mathcal{H}_i$ is a hyperplane, we let $\mathcal{H}_i = V(L_i)$, where L_i is a linear form not involving X_1 and X_2 . We claim that $Q_i \notin \mathcal{H}_i$.

Proof of the claim. We prove the claim by contradiction. If $Q_i \in \mathcal{H}_i$, then $\mathbb{X} \subseteq \mathcal{H}_i$. This implies that $I(\mathcal{H}_i) \subseteq I(\mathbb{X}) \subseteq F^\perp$. This yields $L_i \in F^\perp$, a contradiction. $\square$

Let $L_i = \sum_{j=1}^n \alpha_{ij} Y_j$, and consider the points of $\mathbb{X}$ not on $\mathcal{H}_i = V(L_i)$, that is, the points $\mathbb{X} \cap \Lambda_i$. We have

$$\begin{aligned} F^\perp : I(\mathcal{H}_i) &= F^\perp : (\alpha_{i1} Y_1 + \cdots + \alpha_{in} Y_n) = ((\alpha_{i1} Y_1 + \cdots + \alpha_{in} Y_n) \circ F)^\perp \\ &= (2x_1 x_2 (\alpha_{i1} y_1 + \cdots + \alpha_{in} y_n))^\perp \end{aligned}$$

where the second equality holds by [12, Lemma 3.2]. After a change of coordinates $G = x_1 x_2 L_i$ is a monomial and [13, Proposition 3.1] yields $\text{Rk } G = 4$. Note that

$$I(\mathbb{X} \cap \Lambda_i) = I(\mathbb{X}) : I(\mathcal{H}_i) \subseteq F^\perp : I(\mathcal{H}_i) = G^\perp.$$

Therefore, each of the n planes $\Lambda_i = \langle \ell, Q_i \rangle$ contains at least 4 points of $\mathbb{X}$. Note that F is e-computable with $I = (Y_1, \dots, Y_n)$ and $t = Y_1 + \cdots + Y_n$, and moreover $V(Y_1, \dots, Y_n)$ is the only line in the intersection of any two planes and that, by Remark 2.1.3, $V(Y_1, \dots, Y_n) \subseteq \mathcal{F}_F$. Hence we obtain at least $4n$ points in the union of the n planes. Since $|\mathbb{X}| = 4n$, each plane Λ_i contains exactly 4 points.

To complete the proof of this proposition we need to show that each of these 4 points lie exactly on 2 lines passing through Q_i . This is done in Remark 2.3.4. $\square$

Remark 2.3.2. In the proof of Proposition 2.3.1, we proved that Equation (2.3.6) is an equality. Hence, equality holds in Equations (2.3.5) and (2.3.2). Thus we proved that $\text{HF}(\mathbb{X}, 2) = 3n + 1$.

We now determine the ideal of minimal apolar sets to F .

Proposition 2.3.3. Let $F = x_1 x_2 (y_1^2 + \cdots + y_n^2)$ with $n \geq 2$. If $\mathbb{X}$ is a minimal apolar set of F , then there exists $\sigma \in O(n)$ such that ideal $I(\sigma(\mathbb{X}))$ is given by

$$I(\sigma(\mathbb{X})) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X_1^2 + \sum_{i=2}^n \alpha_i (Y_1^2 - Y_i^2), X_2^2 + \sum_{i=2}^n \beta_i (Y_1^2 - Y_i^2))$$

where $\alpha_i, \beta_i \neq 0$ and $\sum_{i=2}^n \alpha_i \neq 0, \sum_{i=2}^n \beta_i \neq 0$.

Proof. Let $\mathbb{Y}$ be the projection of $\mathbb{X}$ on $V(X_1, X_2)$ from the line $V(Y_1, \dots, Y_n)$. By the proof of Theorem 2.3.1(ii) we have $\mathbb{Y} = \{Q_1, \dots, Q_n\}$ is a set of n points in general position. Moreover,

by Lemma 2.1.6 we have $\mathbb{Y}$ is apolar to $G = (y_1^2 + \cdots + y_n^2)$. Hence by Lemma 2.1.10 we obtain an orthogonal matrix O which acts on S and fixes F , and takes $\mathbb{Y}$ to $\{e_1, \dots, e_n\}$. Hence we may assume $I(\mathbb{Y}) \supseteq (Y_i Y_j \mid 1 \leq i < j \leq n)$. Hence $(Y_i Y_j \mid 1 \leq i < j \leq n) \subseteq I(\mathbb{X})_2$. By Remark 2.3.2 we have $\dim I(\mathbb{X})_2 = \binom{n}{2} + 2$. Hence there exist quadratic forms $\mathcal{Q}_1, \mathcal{Q}_2 \in I(\mathbb{X})_2$ such that

$$I(\mathbb{X})_2 = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, \mathcal{Q}_1, \mathcal{Q}_2).$$

Since $\mathcal{Q}_1, \mathcal{Q}_2 \in F^\perp$, and $\dim I(\mathbb{Y})_2 = \binom{n}{2}$, $\mathcal{Q}_1, \mathcal{Q}_2$ has at least one of the coefficient of X_1^2, X_2^2 non-zero. Thus we may assume

$$\mathcal{Q}_1 = X_1^2 + \sum_{i=2}^n \alpha_i (Y_1^2 - Y_i^2), \quad \mathcal{Q}_2 = X_2^2 + \sum_{i=2}^n \beta_i (Y_1^2 - Y_i^2).$$

We claim that $\alpha_i \neq 0, \beta_i \neq 0$ for $2 \leq i \leq n$ and $\sum_{i=2}^n \alpha_i, \sum_{i=2}^n \beta_i \neq 0$.

Proof of the claim. By contradiction assume $\alpha_i = 0$ for some i and consider the plane defined by $\mathcal{P}_i = (Y_1, \dots, \hat{Y}_i, \dots, Y_n)$. Note that, by Proposition 2.3.1 (ii), $\overline{I(\mathbb{X})} \subset T/\mathcal{P}_i$ is an ideal of 4 points. Moreover, if $\alpha_i = 0$, then $X_1 \in \overline{I(\mathbb{X})}$. Hence, the Hilbert function of $\text{HF}(T/\overline{I(\mathbb{X})}, 2) \leq 2$. This is a contradiction. Similar arguments can be repeated to complete the proof. $\square$

Hence, we write

$$I(\mathbb{X}) = \bigcap_{i=1}^n (Y_1, Y_2, \dots, \hat{Y}_i, \dots, Y_n, \mathcal{Q}_1, \mathcal{Q}_2) + J,$$

where J is generated in degree at least three in $I(\mathbb{X})$.

Consider

$$\bar{\mathcal{Q}}_1, \bar{\mathcal{Q}}_2 \in \frac{\mathbb{C}[X_1, X_2, Y_1, \dots, Y_n]}{(Y_1, \dots, \hat{Y}_i, \dots, Y_n)} \simeq \mathbb{C}[X_1, X_2, Y_i],$$

and note that $\bar{\mathcal{Q}}_1 = X_1^2 - k_1 Y_i^2$, $\bar{\mathcal{Q}}_2 = X_2^2 - k_2 Y_i^2$ with $k_1, k_2 \neq 0$. In particular, $(\bar{\mathcal{Q}}_1, \bar{\mathcal{Q}}_2)$ is an ideal of four points. By Proposition 2.3.1 we now show that $(\bar{\mathcal{Q}}_1, \bar{\mathcal{Q}}_2) + \bar{J}$ is the ideal of the same four points. Thus, $(\bar{\mathcal{Q}}_1, \bar{\mathcal{Q}}_2) \supseteq \bar{J}$ and hence

$$(Y_1, Y_2, \dots, \hat{Y}_i, \dots, Y_n, \mathcal{Q}_1, \mathcal{Q}_2) \supseteq J$$

for all i . The proof is now completed. $\square$

Remark 2.3.4. We can now complete the proof of Proposition 2.3.1 part (ii) final part of showing that 4 points on $\mathbb{X} \cap \Lambda_i$ lie exactly on 2 lines passing through Q_i with exactly two points on each line. Note that, in the proof of Proposition 2.3.3 we have only used the fact that $\mathbb{X} \cap \Lambda_i$ is

a set of 4 points from Proposition 2.3.1 (ii). In particular, we want to show that the four points in each of the planes Λ_i lie on a pair of lines passing through the point Q_i . After a change of coordinates, we may assume Q_i to be the coordinate point e_i , and we may assume the ideal of the four points to be

$$(Y_1, Y_2, \dots, \hat{Y}_i, \dots, Y_n, Q_1, Q_2).$$

Since this ideal contains a degree two polynomial without Y_i , this is enough to conclude.

We now determine the Hilbert function of any minimal apolar set to F .

Proposition 2.3.5. *If $F = x_1x_2(y_1^2 + y_2^2 + \dots + y_n^2)$, then the Hilbert function of any minimal apolar set $\mathbb{X}$ is*

i	0	1	2	3	4	$\dots$
$\text{HF}(\mathbb{X}, i)$	1	$n + 2$	$3n + 1$	$4n$	$4n$	$4n$

(2.3.7)

Proof. By Proposition 2.3.3, after a change of coordinates, we have $I(\mathbb{X}) = (\{Y_iY_j\}_{1 \leq i < j \leq n}, Q_1, Q_2)$ where $Q_1 = X_1^2 + \sum_{i=2}^n \alpha_i(Y_1^2 - Y_i^2)$, and $Q_2 = X_2^2 + \sum_{i=2}^n \beta_i(Y_1^2 - Y_i^2)$. We consider the lexicographic ordering $X_1 > X_2 > Y_1 > \dots > Y_n$ on $\mathbb{C}[X_1, X_2, Y_1, \dots, Y_n]$ and we compute a Gröbner basis G of $I(\mathbb{X})$. In particular, we will show that $G = (\{Y_iY_j\}_{1 \leq i < j \leq n}, Q_1, Q_2)$ is such a basis and this is enough to prove the result. Hence G is a Gröbner basis of $I(\mathbb{X})$. Therefore the initial ideal of $I(\mathbb{X})$ is given by

$$\text{in}_>(I(\mathbb{X})) = (\{Y_iY_j\}_{1 \leq i < j \leq n}, X_1^2, X_2^2).$$

Moreover, $\text{HF}(T/I(\mathbb{X})) = \text{HF}(T/\text{in}_>I(\mathbb{X}))$ and $\text{HF}(T/\text{in}_>I(\mathbb{X}))$ is as in Equation 2.3.7.

To prove that G is Gröbner basis we prove that each S-polynomials is zero mod G . Let $f_{i,j} = Y_iY_j$ and note that the S-polynomial of two monomials is automatically zero. Thus, we only need to consider the following two cases. For any $1 \leq i, j \leq n$ with $i \neq j$, we have

$$\begin{aligned} S(f_{i,j}, Q_1) &= \frac{Y_iY_jX_1^2}{Y_iY_j}(Y_iY_j) - \frac{Y_iY_jX_1^2}{X_1^2}Q_1 \\ &= Y_iY_j \sum_{k=2}^n \alpha_k(Y_1^2 - Y_k^2) \\ &= 0 \mod G. \end{aligned}$$

Similarly, $S(f_{i,j}, Q_2) = 0 \pmod{G}$. Moreover,

$$\begin{aligned}
S(Q_1, Q_2) &= \frac{X_1^2 X_2^2}{X_1^2} Q_1 - \frac{X_1^2 X_2^2}{X_2^2} Q_2 \\
&= X_2^2 \left(\sum_{i=2}^n \alpha_i (Y_1^2 - Y_i^2) \right) - X_1^2 \left(\sum_{i=2}^n \beta_i (Y_1^2 - Y_i^2) \right) \\
&= X_2^2 \left(\sum_{i=2}^n \alpha_i (Y_1^2 - Y_i^2) \right) - X_1^2 \left(\sum_{i=2}^n \beta_i (Y_1^2 - Y_i^2) \right) + X_1^2 X_2^2 - X_1^2 X_2^2 \\
&= X_2^2 Q_1 - X_1^2 Q_2 \\
&= 0 \pmod{G}.
\end{aligned}$$

The proof is now completed. $\square$

We can now characterize which points appear in any minimal apolar sets to F .

Theorem 2.3.6. *If $F = x_1 x_2 (y_1^2 + y_2^2 + \cdots + y_n^2)$, then the forbidden locus of F*

$$\mathcal{F}_F = V(X_1 X_2) \cup V(Y_1^2 + \cdots + Y_n^2).$$

Proof. By Proposition 2.3.3 we have $I(\mathbb{X}) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X_1^2 + \sum_{i=2}^n \alpha_i (Y_1^2 - Y_i^2), X_2^2 + \sum_{i=2}^n \beta_i (Y_1^2 - Y_i^2))$ up to a change of coordinates. First we show that $\text{RHS} \subseteq \text{LHS}$.

Let $P \in \mathcal{W}_F$ and assume that $P \in V(X_1 X_2)$. We will prove that this is a contradiction. If $X_1(P) = 0$ or $X_2(P) = 0$, then, by the shape of the ideal, we get that $P = [0 : 0 : \cdots : 0]$. Thus, $V(X_1 X_2) \subseteq \mathcal{F}_F$.

Now, let $Q = [a : b : c_1 : \cdots : c_n] \in \mathcal{W}_F$ and assume by contradiction that $Q \in V\left(\sum_{i=1}^n Y_i^2\right)$. Let $\mathbb{X}$ be a minimal apolar set containing Q . Following Proposition 2.3.1 we consider $\mathbb{Y}$ the projection of $\mathbb{X}$ from $V(Y_1, \dots, Y_n)$ to $V(X_1, X_2)$. Since $|\mathbb{Y}| = n$, we obtain $\mathbb{Y}$ is a minimal apolar set of $G = y_1^2 + \cdots + y_n^2$ by Lemma 2.1.6. Note that, by Remark 2.1.3, the image of Q under the projection exists and it is $[0 : 0 : c_1 : \cdots : c_n] \in \mathcal{W}_G$. This is a contradiction by Proposition [15, Proposition 3.1] since $\mathcal{F}_G = V\left(\sum_{i=1}^n Y_i^2\right)$. Hence, $V(Y_1^2 + \cdots + Y_n^2) \subseteq \mathcal{F}_F$.

Now we prove that equality holds. Let $Q = [a : b : c_1 : c_2 : \cdots : c_n]$ such that $ab \neq 0$ and $\sum_{i=1}^n c_i^2 \neq 0$. We want to show that $Q \in \mathcal{W}_F$. Let Q' be the projection of Q from $V(Y_1, \dots, Y_n)$ to $V(X_1, X_2)$ given by $Q' = [c_1 : c_2 : \cdots : c_n]$. By Proposition [15, Proposition 3.1] we get $Q' \in \mathcal{W}_G$ where $G = (y_1^2 + \cdots + y_n^2)$. Hence there exists $P'_2, \dots, P'_n$ such that

$\mathbb{Y} = \{Q', P'_2 \dots P'_n\}$ is a minimal apolar set of G . By Lemma 2.1.10 we obtain an orthogonal matrix N with the normalized coordinates of Q' in the first column. Therefore the matrix

$$M = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & N \end{bmatrix}$$

fixes F . Consider the minimal apolar set of F given by the following ideal:

$$I(\mathbb{X}) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X_1^2 - Y_1^2 - \dots - Y_{n-1}^2 + (n-1)Y_n^2, X_1^2 - X_2^2).$$

The action of M takes $\mathbb{X}$ into another minimal apolar set of F . Note that the point $[1, 1, 1, 0, \dots, 0] \in \mathbb{X}$ is mapped to Q by this action. Hence, $Q \in \mathcal{W}_F$. □

We now consider the projections of minimal apolar sets to F on the line $V(X_1, X_2)$.

Proposition 2.3.7. *If $\mathbb{W}$ is the projection of $\mathbb{X}$ from $V(X_1, X_2)$ on $V(Y_1, \dots, Y_n)$, then $\mathbb{W}$ is a set of $2k$ points with $k \leq n$.*

Proof. By Lemma 2.1.5, we have $I(\mathbb{W}) = (Y_1, \dots, Y_n) + I(\mathbb{X}) \cap \mathbb{C}[X_1, X_2]$. To prove that the maximal number of points is $2n$, it is enough to prove that $(I(\mathbb{X})_{2n} \cap \mathbb{C}[X_1, X_2]) \neq 0$. By Proposition 2.3.3 we have

$$\begin{aligned} I(\mathbb{X}) &= (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X_1^2 + \sum_{i=2}^n \alpha_{1i}(Y_1^2 - Y_n^2), X_2^2 + \sum_{i=2}^n \alpha_{2i}(Y_1^2 - Y_i^2)) \\ &= \bigcap_{i=1}^n (Y_1, \dots, \hat{Y}_i, \dots, Y_n, X_1^2 - k_{1i}Y_i^2, X_2^2 - k_{2i}Y_i^2) \end{aligned} \quad (2.3.8)$$

where $k_{ji} = \alpha_{ji}$ for $1 \leq j \leq 2$ and $2 \leq i \leq n$, and $k_{j1} = \sum_{i=2}^n \alpha_{ji}$ for $1 \leq j \leq 2$. Hence

$\prod_{i=1}^n (k_{2i}X_1^2 - k_{1i}X_2^2) \in I(\mathbb{X})$. Thus the first part of the proof is completed.

Now we prove that only an even number of points can occur in $\mathbb{W}$. Note that formula 2.3.8 gives a primary decomposition of $I(\mathbb{X})$. In particular, to understand the projection $\mathbb{W}$, we need to study the common roots of the polynomials $X_1^2 - k_{1i}X_2^2$ for $1 \leq i \leq n$. Any pair of these polynomials either has two common roots or no common root at all. Hence, the result follows. □

Theorem [I. B](#) is now proved by using Propositions [2.3.1](#), [2.3.3](#), [2.3.5](#), Theorem [2.3.6](#), and Proposition [2.3.7](#).

The previous Proposition [2.3.7](#) shows that the projection $\mathbb{W}$, unlikely the projection $\mathbb{Y}$, is not completely determined. In the remark that follows we give more details on $\mathbb{W}$.

Remark 2.3.8. Keeping the same notation as in Proposition [2.3.1](#), we note the following:

(i) If we choose $\alpha_{1i} = \alpha_{2i}$ for all $2 \leq i \leq n$, then $X_1^2 - X_2^2 \in I(\mathbb{X})$. Thus, $\mathbb{W}$ contains exactly 2 points.

(ii) If $\frac{\alpha_{1i}}{\alpha_{2i}} \neq \frac{\alpha_{1j}}{\alpha_{2j}}$ for any $2 \leq i, j \leq n$, $i \neq j$ such that $\frac{\sum_{k=2}^n \alpha_{1k}}{\sum_{k=2}^n \alpha_{2k}} \neq \frac{\alpha_{1i}}{\alpha_{2i}}$ for any $2 \leq i \leq n$, we obtain exactly $2n$ points in $\mathbb{W}$. For example, choose $\alpha_{1i} = i - 1$ for $2 \leq i \leq n - 1$, $\alpha_{1n} = e^n$ and $\alpha_{2i} = 1$ for all $2 \leq i \leq n$.

(iii) For $n = 3$ we can not get exactly 4 points in $\mathbb{W}$. Indeed, note that if $\frac{\alpha_2}{\beta_2} \neq \frac{\alpha_3}{\beta_3}$ implies

$$\frac{\alpha_2 + \alpha_3}{\beta_2 + \beta_3} \neq \frac{\alpha_2}{\beta_2} \quad \text{and} \quad \frac{\alpha_2 + \alpha_3}{\beta_2 + \beta_3} \neq \frac{\alpha_3}{\beta_3}$$

This implies $\mathbb{W}$ has exactly 6 points. Further, if $\frac{\alpha_2}{\beta_2} = \frac{\alpha_3}{\beta_3}$, then we obtain exactly 2 points in $\mathbb{W}$.

(iv) For $n \geq 4$, we can choose $\frac{\alpha_i}{\beta_i}$ such that $\mathbb{W}$ has 4 points. i.e. choose $\frac{\alpha_2}{\beta_2} \neq \frac{\alpha_3}{\beta_3}$, and $\frac{\alpha_3}{\beta_3} = \frac{\alpha_i}{\beta_i}$ for all $i \geq 4$ with $\sum_{i=2}^{n-1} \alpha_i = \sum_{i=2}^{n-1} \beta_i = 0$.

Example 2.3.9. Let $F = (x_1^2 + x_2^2)(y_1^2 + y_2^2 + y_3^2)$. By Remark [2.3.8](#) we have that the projection of any minimal apolar set $\mathbb{X}$ from $V(X_1, X_2)$ to $V(Y_1, Y_2, Y_3)$ is either a set of 2 points or a set of 6 points, while the projection of $\mathbb{X}$ from $V(Y_1, Y_2, Y_3)$ to $V(X_1, X_2)$ is always a set of 3 points. Hence, by Theorem [I. B](#) we obtain that any minimal apolar set $\mathbb{X}$ to F has one of the following shapes described in Figure [2.1](#) and Figure [2.2](#). Red dots denote the points in $\mathbb{X}$ in both pictures.

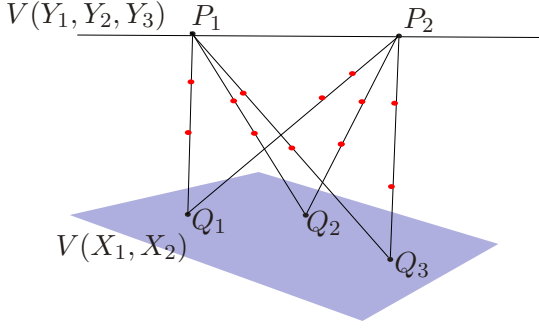


Figure 2.1: Minimal apolar set whose projection from $V(X_1, X_2)$ to $V(Y_1, Y_2, Y_3)$ having 2 points.

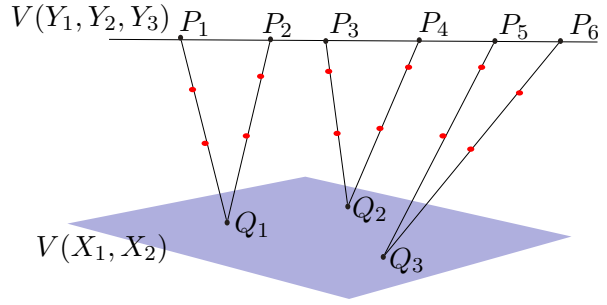


Figure 2.2: Minimal apolar set whose projection from $V(X_1, X_2)$ to $V(Y_1, Y_2, Y_3)$ having 6 points.

2.4 The forms $(x_1^2 + x_2^2 + \cdots + x_m^2)(y_1^2 + y_2^2 + \cdots + y_n^2)$

In this section, we provide a new lower and an upper bound for the rank of the forms $(x_1^2 + x_2^2 + \cdots + x_m^2)(y_1^2 + y_2^2 + \cdots + y_n^2)$.

Proposition 2.4.1. *Let $n \geq 2$. If we let $F = (x_1^2 + x_2^2 + x_3^2)(y_1^2 + \cdots + y_n^2)$, then $\text{Rk } F \leq 6n$.*

Proof. Consider the ideal

$$I(\mathbb{X}) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, X_1 X_2, X_1 X_3, X_2^2 - X_3^2, X_1^2 - X_2^2 + Y_1^2 + \cdots + Y_{n-1}^2 - (n-1)Y_n^2).$$

Using a description of $F^\perp$ given in Lemma 2.1.7, it is clear that $I(\mathbb{X}) \subset F^\perp$. Moreover

$$I(\mathbb{X}) = \left(\bigcap_{i=1}^n (Y_1, \dots, \hat{Y}_i, \dots, Y_n, X_1, X_2^2 - X_3^2, -X_2^2 + k_i Y_i^2) \right) \\ \bigcap \left(\bigcap_{i=1}^n (Y_1, \dots, \hat{Y}_i, \dots, Y_n, X_2, X_3, X_1^2 + k_i Y_i^2) \right)$$

where $k_i = 1$ for $1 \leq i \leq n-1$ and $k_n = n-1$. Thus, $\mathbb{X}$ is an ideal of $6n$ points apolar to F .

Hence, by the Apolarity Lemma 2.1.1 the result is proved. $\square$

Proposition 2.4.2. *Let $m, n \geq 2$. If we let $F = (x_1^2 + \cdots + x_m^2)(y_1^2 + \cdots + y_n^2)$, then $\text{Rk } F \leq 2mn$.*

Proof. Let $q = y_1^2 + \cdots + y_n^2$. We divide the proof in two cases: m even and m odd.

Case (i) $m = 2k$ for some $k \in \mathbb{N}$. Then we write

$$(x_1^2 + \cdots + x_m^2)q = (x_1^2 + x_2^2)q + \cdots + (x_{2k-1}^2 + x_{2k}^2)q.$$

Therefore by Proposition [2.4.4](#) we have $\text{Rk}(x_i^2 + x_j^2)q = \text{Rk}(x_i + \iota x_j)(x_i - \iota x_j)q = 4n$.

Hence we get

$$\text{Rk } F \leq 4nk = 2n \cdot 2k = 2mn.$$

Case (ii) $m = 2k + 1$ for some $k \in \mathbb{N}$. Then we write

$$(x_1^2 + \cdots + x_m^2)q = (x_1^2 + \cdots + x_{2k-2}^2)q + (x_{2k-1}^2 + x_{2k}^2 + x_{2k+1}^2)q.$$

Therefore, by case (i) and Proposition [2.4.1](#), we get

$$\text{Rk } F \leq 2(2k - 2)n + 6n = 2n \cdot (2k + 1) = 2mn.$$

The proof is completed. □

Theorem 2.4.3. *Let $n \geq m \geq 3$. If we let $F = (x_1^2 + \cdots + x_m^2)(y_1^2 + \cdots + y_n^2)$, then*

$$n(m + 2) < \text{Rk } F \leq 2mn.$$

Proof. Note that the upper bound comes from Proposition [2.4.2](#), thus we only need to prove the lower bound. Using Lemma [2.1.7](#) we get

$$F^\perp = (\{X_i X_j\}_{1 \leq i < j \leq m}, \{X_1^2 - X_i^2\}_{2 \leq i \leq m}, \{Y_i Y_j\}_{1 \leq i < j \leq n}, \{Y_1^2 - Y_i^2\}_{2 \leq i \leq n}).$$

Set $I = (Y_1, \dots, Y_n)$, $t = \alpha_1 Y_1 + \cdots + \alpha_n Y_n$ for $0 \neq \alpha_i \in \mathbb{C}$, $1 \leq n$, and $J = (F^\perp : I) + (t)$.

Thus, we get

$$J = (\{X_i X_j\}_{1 \leq i < j \leq m}, \{X_1^2 - X_i^2\}_{2 \leq i \leq m}, \{Y_i Y_j\}_{1 \leq i < j \leq n}, Y_i^2\}_{1 \leq i \leq n}) + (t).$$

Therefore

i	0	1	2	3	4
$\text{HF}(T/J, i)$	1	$m + n - 1$	$m(n - 1) + 1$	$n - 1$	0

and thus $\sum_{i \geq 0} \text{HF}(T/J, i) = n(m + 2)$. Hence, by [\[12, Theorem 3.3\]](#), we get

$$\text{Rk } F \geq \sum_{i \geq 0} \text{HF}(T/J, i) = n(m + 2).$$

We now prove by contradiction that $\text{Rk } F > n(m + 2)$. We assume that $\text{Rk } F = n(m + 2)$ and we let $\mathbb{X}$ being a minimal apolar subset of F . Note that F is e -computable using $I = (Y_1, \dots, Y_n)$ and $t = \sum_{i=1}^n \alpha_i Y_i$ for $0 \neq \alpha_i \in \mathbb{C}$. Thus, Remark [2.1.3](#) yields $V(Y_1, Y_2, \dots, Y_n) \subseteq \mathcal{F}_F$.

Let $V = F_2^\perp \cap \mathbb{C}[Y_1, \dots, Y_n]_2$. Then $\dim V = \binom{n}{2} + n - 1$. If we let $\dim V/I(\mathbb{X}) = \dim V - \alpha$, then $\dim I(\mathbb{X})_2 \cap \mathbb{C}[Y_1, \dots, Y_n]_2 = \alpha$. Thus, by Lemma 2.1.4 we have

$$\text{HF}(\mathbb{X}, 2) \leq n(m+2) - \dim V/I(\mathbb{X}).$$

Therefore

$$\dim I(\mathbb{X})_2 \geq \binom{n+m+1}{2} - n(m+2) + \dim V/I(\mathbb{X}).$$

Applying Grassmann's formula, we get

$$\begin{aligned} \alpha &= \dim I(\mathbb{X})_2 \cap \mathbb{C}[Y_1, \dots, Y_n]_2 = \dim I(\mathbb{X})_2 + \dim V - \dim(V + I(\mathbb{X})_2) \\ &\geq \dim I(\mathbb{X})_2 + \dim V - \dim F_2^\perp \\ &\geq \binom{n+m+1}{2} - n(m+2) + \dim V/I(\mathbb{X}) + \dim V - \dim F_2^\perp \\ &\geq 2\binom{n}{2} - \alpha. \end{aligned} \tag{2.4.1}$$

and thus, $\alpha \geq \binom{n}{2}$.

We now show that $\alpha = \binom{n}{2}$. Note that $(X_1, X_2, \dots, X_m) + I(\mathbb{X}) \cap \mathbb{C}[Y_1, \dots, Y_n]$ is the ideal of the projection of $\mathbb{X}$ onto $V(X_1, X_2, \dots, X_m)$, we call this projection $\mathbb{Y}$. Since $\text{HF}(\mathbb{Y}, 1) = n$, we get $\text{HF}(\mathbb{Y}, 2) \geq n$. Thus,

$$\text{HF}(\mathbb{Y}, 2) = \binom{n+1}{2} - \dim I(\mathbb{X})_2 \cap \mathbb{C}[Y_1, \dots, Y_n]_2 \geq n$$

implies

$$\alpha \leq \binom{n+1}{2} - n = \binom{n}{2}. \tag{2.4.2}$$

Therefore, by Equations (2.4.1), and (2.4.2), we have

$$\dim I(\mathbb{X})_2 \cap \mathbb{C}[Y_1, \dots, Y_n]_2 = \binom{n}{2}.$$

Therefore, $\text{HF}(\mathbb{Y}, 2) = n$ and thus, $I(\mathbb{Y})$ is an ideal of n points. Therefore the apolar set $\mathbb{X}$ is contained in n many m -dimensional linear spaces, each passing through $V(Y_1, \dots, Y_n)$. Similarly as in the proof of Proposition 2.2.1, let $\mathbb{Y} = \{Q_1, \dots, Q_n\}$, and $\mathcal{H}_\ell = \langle \ell, Q_1, \dots, \hat{Q}_i, \dots, Q_n \rangle$ where $\ell = V(Y_1, \dots, Y_n)$. Note that $\mathcal{H}_i$ is a hyperplane, hence $I(\mathcal{H}_i) = (L_i)$ for some $L_i = \alpha_i 1Y_1 + \dots + \alpha_i nY_n$. Indeed,

$$\begin{aligned} F^\perp : (\alpha_{i1}Y_1 + \dots + \alpha_{in}Y_n) &= ((\alpha_{i1}Y_1 + \dots + \alpha_{in}Y_n) \circ F)^\perp \\ &= ((x_1^2 + x_2^2 + \dots + x_m^2)(\alpha_{i1}y_1 + \dots + \alpha_{in}y_n))^\perp \end{aligned}$$

where the first equality holds by [12] Lemma 3.2]. Note that $G = (x_1^2 + x_2^2 + \cdots + x_m^2)(\alpha_1 y_1 + \cdots + \alpha_n y_n)$ is such that $\text{Rk } G = 2m$. Again, by Proposition 2.4.4 F is e-computable with $I = (Y_1, \dots, Y_n)$ and $t = Y_1 + \cdots + Y_n$. Hence by Remark 2.1.3 we have $V(Y_1, Y_2, \dots, Y_n) \subseteq \mathcal{F}_F$, we obtain $\mathbb{X}$ contains $2mn$ distinct points. Hence a contradiction since $2mn > n(m+2)$. $\square$

Next, we prove the Waring rank of the forms of the type $F = x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} (y_1^b + \cdots + y_n^b)$, where $m, n \geq 1$, $b \geq 2$, and $a_1 \leq a_2 \leq \cdots \leq a_m$ with $a_1 + 1 \geq b$. Note that this includes the case $F = (x_1^2 + x_2^2)(y_1^2 + \cdots + y_n^2)$. To understand the structure of minimal apolar sets of such forms, the computation of rank was crucial.

Proposition 2.4.4. *Let $F = x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} (y_1^b + \cdots + y_n^b)$, where $m, n \geq 1$, $b \geq 2$, and $a_1 \leq a_2 \leq \cdots \leq a_m$ with $a_1 + 1 \geq b$. Then*

$$\text{Rk } F = \left(\prod_{i=1}^m (a_i + 1) \right) n.$$

Proof. We use e-computability to obtain a lower bound. Consider $I = (Y_1, Y_2, \dots, Y_n)$ and $t = \sum_{i=1}^n \alpha_i Y_i$. Let $\tilde{I} = F^\perp : I + (t)$. Then

$$\tilde{I} = (\{X_i^{a_i+1}\}_{1 \leq i \leq n}, \{Y_i Y_j\}_{1 \leq i < j \leq n}, \{Y_i^b\}_{1 \leq i \leq n}, \sum_{i=1}^n \alpha_i Y_i). \quad (2.4.3)$$

We want to compute the Hilbert function of $T/\tilde{I}$. For $l \geq 0$, and $l+1 < b$, we have

$$\begin{aligned} \text{HF}(T/\tilde{I}, l+1) &= (n-1)(\text{No. of linearly independent degree } l \text{ monomials in } T/\tilde{I} \\ &\quad \text{involving only } x'_i\text{'s}) + (\text{No. of linearly independent degree } l+1 \text{ monomials in } T/\tilde{I} \\ &\quad \text{involving only } x'_i\text{'s}). \end{aligned} \quad (2.4.4)$$

Let $\mathbf{a} = (a_1, a_2, \dots, a_m)$. Define

$$C(m, s, \mathbf{a}) = |\{(l_1, l_2, \dots, l_m) \mid \sum_{i=1}^m l_i = s \text{ and } l_i \leq a_i\}|.$$

Here $C(m, s, \mathbf{a})$ counts the number of linearly independent monomials of degree s in $T/\tilde{I}$ involving only x'_i 's. By Equation (2.4.4), for all $l \geq 0$,

$$\text{HF}(T/\tilde{I}, l+1) = C(m, l, \mathbf{a})(n-1) + C(m, l+1, \mathbf{a}). \quad (2.4.5)$$

Further, $\text{HF}(T/\tilde{I}, (\sum_{i=1}^m a_i) + 1) = 0$. Let $\alpha = \sum_{i=1}^m a_i$. We have

$$\begin{aligned}
\sum_{l=0}^{\alpha+1} \text{HF}(T/\tilde{I}, l) &= 1 + \sum_{l=1}^{\alpha+1} \text{HF}(T/\tilde{I}, l) = 1 + \sum_{l=0}^{\alpha} \text{HF}(T/\tilde{I}, l+1) \\
&= 1 + \sum_{l=0}^{\alpha} (C(m, l, \mathbf{a})(n-1) + C(m, l+1, \mathbf{a})) \quad (\text{by Equation (2.4.5)}) \\
&= n \sum_{l=0}^{\alpha} C(m, l, \mathbf{a}) = n \prod_{i=1}^m (a_i + 1)
\end{aligned} \tag{2.4.6}$$

Therefore, by Equation (2.4.6) we have

$$\text{Rk } F \geq \sum_{l \geq 0} \text{HF}(T/\tilde{I}, l) = n \prod_{i=1}^m (a_i + 1), \tag{2.4.7}$$

where the first inequality holds by [12, Theorem 3.3]. Further, the ideal

$$I(\mathbb{X}) = (\{Y_i Y_j\}_{1 \leq i < j \leq n}, Q_1, Q_2, \dots, Q_m)$$

where $Q_i = X_i^{a_i+1} + (Y_1^{a_i+1-b} + \dots + Y_n^{a_i+1-b})((-n+1)Y_1^b + Y_2^b + \dots + Y_n^b)$ is contained in $F^\perp$. We have

$$I(\mathbb{X}) = \bigcap_{i=1}^n (Y_1, Y_2, \dots, \hat{Y}_i \dots Y_n, q_1, q_2, \dots, q_m),$$

where $q_j = X_j^{a_j+1} + Y_i^{a_i+1}$ for all $1 \leq j \leq m$, for all $2 \leq i \leq n$, and for $i = 1$, for all $1 \leq j \leq m$, $q_j = X_j^{a_j+1} + (-n+1)Y_1^{a_1+1}$. Note that over the field of complex numbers, q_j splits into linear forms, giving the primary decomposition of $I(\mathbb{X})$. Hence $I(\mathbb{X})$ is an ideal of $n \prod_{i=1}^m (a_i + 1)$ distinct points. Therefore, by the Apolarity Lemma,

$$\text{Rk } F \leq n \prod_{i=1}^m (a_i + 1). \tag{2.4.8}$$

Therefore, by inequalities (2.4.7) and (2.4.8), we obtain the result. $\square$

Remark 2.4.5. Our guess is that the upper bound is the Waring rank of F in Theorem 2.4.3. We prove $m = 2$ case in Proposition 2.4.4. If the guess is correct, then we have a nice structure for minimal apolar sets of F .

Chapter 3

Waring rank of quasi-monomial complete intersections

In this chapter, first, we compute the Waring rank of the forms F such that

$$F^\perp = (X_0^{a_0} + G(X_0, \dots, X_n), X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n})$$

where $G(X_0, \dots, X_n)$ is a homogeneous form of degree a_0 with $\deg_{X_0} G < a_0$. This is done in Section 3.2. As an application, in Section 3.3 we further compute the Waring rank of binomials whose annihilator ideal is a complete intersection. To obtain lower bounds on the Waring rank, we highly depend on e-computability 2.1.2. Hence, in Section 3.1 we compute the relevant colon ideals.

3.1 Preliminary results

In the rest of the chapter, we consider the standard graded polynomial ring $T = \mathbb{C}[X_0, \dots, X_n]$.

Let $T[t] = \mathbb{C}[t, X_0, X_1, \dots, X_n]$ be the standard graded polynomial ring with the *lexicographic ordering* $t > X_0 > \dots > X_n$. Next, we compute $F^\perp : (X_i)$, $1 \leq i \leq n$, and $F^\perp : (h)$ when $X_0^{a_0} + G = h^m g$ for some forms $h, g \in T$ with $\deg h > 0$ and $m \geq 2$. In the next section, using these colon ideals and Theorem 2.1.2 on e-computability, we obtain lower bounds on the Waring rank of F .

Proposition 3.1.1. *Let $I = (X_0^{a_0} + G, X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n}) \subset T$ be an ideal.*

1. *For any $1 \leq i \leq n$, we have*

$$I : (X_i) = (X_0^{a_0} + G, X_1^{a_1}, \dots, X_i^{a_i-1}, \dots, X_n^{a_n}).$$

2. If $X_0^{a_0} + G = h^m g$ for forms $h, g \in T$ with $\deg h > 0$ and $m \geq 2$, then

$$I : (h) = (h^{m-1}g, X_1^{a_1}, \dots, X_n^{a_n}).$$

Proof. ^[1] We first prove (1). Note that a Gröbner basis of $tI + (1-t)(X_i) \subset T[t]$ is,

$$GB = \{tX_0^{a_0} + tG, tX_1^{a_1}, tX_2^{a_2}, tX_3^{a_3}, \dots, tX_n^{a_n}, (t-1)X_i, X_i(X_0^{a_0} + G), X_iX_1^{a_1}, X_iX_2^{a_2}, \dots, X_i^{a_i}, \dots, X_iX_n^{a_n}\}.$$

Thus by Theorem [B.1](#) we have that

$$I \cap (X_i) = (X_i(X_0^{a_0} + G), X_iX_1^{a_1}, X_iX_2^{a_2}, \dots, X_i^{a_i}, \dots, X_iX_n^{a_n}).$$

The conclusion follows by Theorem [B.2](#). Similarly, to prove (2) we use that

$$GB = \{th^m g, tX_1^{a_1}, tX_2^{a_2}, \dots, tX_n^{a_n}, (t-1)h, h^m g, hX_1^{a_1}, hX_2^{a_2}, \dots, hX_n^{a_n}\}$$

is a Gröbner basis of $tI + (1-t)(h) \subset T[t]$. Thus by Theorem [B.1](#) we get

$$I \cap (h) = (h^m g, hX_1^{a_1}, hX_2^{a_2}, \dots, hX_n^{a_n}).$$

Again the conclusion follows by Theorem [B.2](#). □

Recall that a set of points $\mathbb{X} \subseteq \mathbb{P}^n$ is said to be apolar to F if $I(\mathbb{X}) \subseteq F^\perp$. To prove the existence of some apolar set, we use a geometric approach. We recall a few notation. Let $\mathbb{V} \subset \mathbb{P}^n$ be a projective variety. We denote the homogeneous coordinate ring of $\mathbb{V}$ by $\mathbb{C}[\mathbb{V}]$ and the field of rational functions of $\mathbb{V}$ by $\mathbb{C}(\mathbb{V})$, that is, the field of fractions of $\mathbb{C}[\mathbb{V}]$. Let $f : \mathbb{V} \rightarrow \mathbb{W}$ be a map of varieties. Recall that f is said to be a *finite* map if $\mathbb{C}[\mathbb{V}]$ is integral over $\mathbb{C}[\mathbb{W}]$.

Lemma 3.1.2. *If $f = X_0^{a_0} + G \in T$ is a homogeneous irreducible polynomial of degree a_0 , and*

$$\pi_p : V(f) \rightarrow \mathbb{P}^{n-1}$$

is the map defined by the projection of $V(f)$ from the point $[1 : 0 : \dots : 0]$ to $\mathbb{P}^{n-1}$, then the pullback $\pi_p^*(\mathbb{C}(\mathbb{P}^{n-1})) \subseteq \mathbb{C}(V(f))$ is a finite separable field extension.

Proof. We have $\mathbb{C}[V(f)] = \mathbb{C}[X_0, \dots, X_n]/(f)$ and $\mathbb{C}[\mathbb{P}^{n-1}] = \mathbb{C}[X_0, \dots, X_n]/(X_0) \cong \mathbb{C}[X_1, \dots, X_n]$. Note that f is a finite map. Indeed, by the properties of integral extension,

¹See Proposition [B.3](#) for a detailed version.

it suffices to show that X_0 is integral over $\mathbb{C}[\mathbb{P}^{n-1}]$. Considering f as a polynomial in X_0 with coefficients from $\mathbb{C}[\mathbb{P}^{n-1}]$, X_0 is integral over $\mathbb{C}[\mathbb{P}^{n-1}]$. Hence $\mathbb{C}[V(f)]$ is finite over $\mathbb{C}[\mathbb{P}^{n-1}]$. Therefore $\mathbb{C}(V(f))$ is a finite field extension over $\mathbb{C}(\mathbb{P}^{n-1})$. Since $\mathbb{C}(\mathbb{P}^{n-1})$ is of characteristic zero, it is a perfect field and every finite extension of a perfect field is separable. Hence the proof. $\square$

The following result describes a nice property of maps as in [3.1.2](#) between normal varieties.

Theorem 3.1.3 (Theorem 2.29 [\[66\]](#)). *Let X, Y be irreducible varieties, Y be normal and $\mathcal{F} : X \rightarrow Y$ be a map. If $\mathcal{F}^*(k(Y)) \subseteq k(X)$ is a finite separable field extension, then the set of unramified points of $\mathcal{F}$ is open and nonempty.*

Next, we recall a classical result: Bertini's Theorem. This result is useful in obtaining an ideal of a certain number of distinct points inside the annihilator ideal.

Theorem 3.1.4 (Corollary 10.9 [\[28\]](#)). *Let $\mathbb{X}$ be a non-singular projective variety over an algebraically closed field of characteristic zero. If δ is a linear system without base points on $\mathbb{X}$, then almost every element of δ considered as a closed subscheme of $\mathbb{X}$ is nonsingular.*

Remark 3.1.5. We note that from [\[28, Remark 10.9.2\]](#) that “projective” is not necessary in the Theorem [3.1.4](#) if the linear system δ is finite dimensional (which is indeed the case in our set up). Moreover, if $\mathbb{X}$ has finitely many singular points, and δ is base point free on $\mathbb{X}$, then by [\[28, Remark 8.18.1\]](#) there exists an element in δ which is smooth on $\mathbb{X} \setminus \{\text{singular points of } \mathbb{X}\}$.

3.2 Rank of quasi-monomial complete intersections

In this section we compute the Waring rank of F such that

$$F^\perp = (X_0^{a_0} + G(X_0, \dots, X_n), X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n})$$

where $G(X_0, \dots, X_n)$ is a homogeneous form of degree a_0 with $\deg_{X_0} G(X_0, \dots, X_n) < a_0$. For the sake of simplicity, we denote $X_0^{a_0} + G(X_0, \dots, X_n)$ by $X_0^{a_0} + G$.

Theorem 3.2.1. *Let $F \in T$ be such that $F^\perp = (X_0^{a_0} + G, X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n})$ where G is a form of degree a_0 with $\deg_{X_0} G < a_0$ and $a_1 \leq a_2 \leq \dots \leq a_n$. Then*

$$\text{Rk } F = \begin{cases} a_0 \prod_{j=2}^n a_j & \text{if } a_1 \leq a_0 \text{ or} \\ & a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ does not have a multiple factor.} \\ \prod_{j=1}^n a_j & \text{if } a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ has a multiple factor.} \end{cases}$$

Proof. We divide the proof in two cases.

Case I: $(a_1 \leq a_0)$ or $(a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ does not have a multiple factor})$.

First, we prove a lower bound on the $\text{Rk } F$ using e-compatibility. Let $J = (X_1)$ and $t = X_1$.

By Proposition [3.1.1\(1\)](#) we get

$$\tilde{I} = (F^\perp : J) + (t) = (X_0^{a_0} + G, X_1, X_2^{a_2}, \dots, X_n^{a_n}).$$

And thus

$$T/\tilde{I} = \frac{\mathbb{C}[X_0, \dots, X_n]}{(X_0^{a_0} + G, X_1, X_2^{a_2}, \dots, X_n^{a_n})}.$$

Therefore, we get

$$\sum_{i \geq 0} \text{HF}(T/\tilde{I}, i) = a_0 \prod_{i=2}^n a_i.$$

Furthermore, Theorem [2.1.2](#) on e-compatibility yields the lower bound

$$\text{Rk } F \geq a_0 \prod_{i=2}^n a_i. \quad (3.2.1)$$

To obtain an upper bound on $\text{Rk } F$ we use the Apolarity Lemma [2.1.1](#). That is, we construct a set of $a_0 \prod_{i=2}^n a_i$ distinct points apolar to F . We do this in two subcases.

Case I(i): We first prove an upper bound for $\text{Rk } F$ if $a_1 \leq a_0$.

Consider the projective variety

$$\mathbb{V} = V(X_2^{a_2} - X_1^{a_2}, \dots, X_n^{a_n} - X_1^{a_n})$$

with $a_i \geq 2$ for $2 \leq i \leq n$. Note that $I(\mathbb{V}) \subseteq F^\perp$. Let S be the singular locus of $\mathbb{V}$. We compute S using the Jacobian criterion. Note that the Jacobian matrix is given by

$$J = \begin{pmatrix} 0 & -a_2 X_1^{a_2-1} & a_2 X_2^{a_2-1} & 0 & \dots & 0 \\ 0 & -a_3 X_1^{a_3-1} & 0 & a_3 X_3^{a_3-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -a_n X_1^{a_n-1} & 0 & 0 & \dots & a_n X_n^{a_n-1} \end{pmatrix}_{(n-1) \times (n+1)}.$$

If $P = [p_0, \dots, p_n] \in \text{Sing}(\mathbb{V})$, then $\text{rank } J(P) \leq n-2$. This means all the determinants of size $(n-1)$ vanishes at P . i.e., for $1 \leq j \leq n$, $D_j(P) = \frac{\prod_{i=1}^n X_i^{a_i-1}}{X_j^{a_j-1}}(P) = 0$. This means $X_l(P) = 0$ for some $2 \leq l \leq n$. Since $P \in \mathbb{V}$, we have $X_l(P) = X_1(P) = X_2(P) = \dots = X_n(P) = 0$. Hence $S = \{[1 : 0 : 0 : \dots : 0]\}$. Let $\mathbb{V}' = \mathbb{V} \setminus S$, and $\delta = (F^\perp)_{a_0}$ be the vector space defining a linear system on $\mathbb{V}'$. i.e., given $v \in F_{a_0}^\perp$ define a divisor $D_v = \{Z \in \mathbb{V}' \mid v(Z) = 0\}$ on $\mathbb{V}'$. We *claim* that the linear system defined by δ on $\mathbb{V}'$ is base point free, that is $\text{BS}(\delta) = \emptyset$.

Proof of the claim. By contradiction, assume that $q = [q_0 : q_1 : \dots : q_n] \in \mathbb{V}'$ is a base point of δ . Since $X_1^{a_0} \in \delta$ we get $q_1 = 0$. This implies $q_1 = \dots = q_n = 0$ because $q \in \mathbb{V}$. Thus a contradiction since $[1 : 0 : \dots : 0] \notin \mathbb{V}'$. Hence the claim. $\square$

By Theorem [3.1.4](#) we know that a general element of δ is smooth on $\mathbb{V}'$. Thus there exists $H \in F_{a_0}^\perp$ such that

$$\mathbb{X} = V(H, X_2^{a_2} - X_1^{a_2}, \dots, X_n^{a_n} - X_1^{a_n})$$

is a smooth set of points. Moreover $\mathbb{X}$ is a set of $a_0 \prod_{i=2}^n a_i$ distinct points with $I(\mathbb{X}) \subseteq F^\perp$. Hence the Apolarity Lemma [2.1.1](#) yields the upper bound

$$\text{Rk } F \leq a_0 \prod_{i=2}^n a_i. \quad (3.2.2)$$

Therefore, by Equations [\(3.2.1\)](#) and [\(3.2.2\)](#), we get the exact value of $\text{Rk } F$.

Case I(ii): To obtain an upper bound for the $\text{Rk } F$ if $a_1 > a_0$ and $(X_0^{a_0} + G)$ does not have a multiple factor, we proceed as follows.

Let $\mathcal{L}_P$ be the set lines in $\mathbb{P}^n$ passing through the point P and $\mathcal{L}'_P \subset \mathcal{L}_P$ be the set of lines that intersect $V(X_0^{a_0} + G)$ in distinct points. Note that $f = X_0^{a_0} + G$ does not have multiple factors, hence we assume f is irreducible. If not, write f as a product of irreducible factors, and we can repeat the argument below on each component separately. Consider the projection map from the point $P = [1 : 0 : 0 : \dots : 0]$

$$\pi_P : V(f) \rightarrow \mathbb{P}^{n-1}. \quad (3.2.3)$$

By Lemma [3.1.2](#) $\pi_P^*(k(\mathbb{P}^{n-1}) \subset k(V(f)))$ is a separable field extension. Hence, by Theorem [\[66, Theorem 2.29\]](#) the set of unramified points of π_P is open and nonempty. Thus the ramification locus of π_P is a proper closed subset; we call this locus Z . Therefore $\mathcal{L}'_P$ is dense and non-empty.

Let us define apolar star to be the union of lines $\mathcal{X}$ passing through the point P defined by the ideal

$$(X_2^{a_2} - w_2 X_1^{a_2}, \dots, X_n^{a_n} - w_n X_1^{a_n}) \subset F^\perp \quad (3.2.4)$$

with $w_i \neq 0$ for all $2 \leq i \leq n$. Let $\mathcal{L}_P'' \subset \mathcal{L}_P$ be the set of all lines containing apolar sets as in Equation (3.2.4). Then $\mathcal{L}_P''$ is dense and non-empty. Indeed, the set of all apolar stars consists of all the lines except the lines passing through the union of the surfaces $\{X_i = 0 \mid 0 \leq i \leq n\}$. Hence the complement is open.

Note that the apolar star $\mathcal{X}$ is completely determined by any lines in it. Moreover, given a line ℓ , there is a unique apolar star containing it. A bad apolar star $\mathcal{X}$ is an apolar star such that $\pi_P(\mathcal{X}) \cap Z \neq \emptyset$. That means there is a line ℓ in $\mathcal{X}$ intersect the ramification locus Z . i.e., $\ell \in \mathcal{X}$ does not intersect $V(f)$ in distinct points. Note that $\mathcal{X}$ is completely determined by ℓ and $\pi_P(\ell) \in Z$. Thus the bad apolar star is contained in a closed subset of $\mathcal{L}_P$ and thus proper. In other terms $\mathcal{L}_P''' \subset \mathcal{L}_P$ the set of lines describing apolar star which is not bad, is open and non-empty. If $\pi_P(\ell) = [1 : w_2 : \cdots : w_n]$ and $\ell \in \mathcal{L}_P' \cap \mathcal{L}_P'' \cap \mathcal{L}_P'''$, then $\mathcal{X}_\ell \cap V(f)$ is a set of distinct $a_0 \prod_{i=2}^n a_i$ points apolar to F .

Case II: $a_1 > a_0$ and $X_0^{a_0} + G$ has a multiple factor.

Let $(X_0^{a_0} + G) = h^m g$ with $m \geq 2$ for some $h, g \in T$ where h is a form of degree $k \geq 1$. Consider $J = (h)$ and $t = h$. Therefore by Proposition 3.1.1(2) we get

$$\tilde{I} = (F^\perp : J) + (t) = (h, X_1^{a_1}, \dots, X_n^{a_n}). \quad (3.2.5)$$

Note that $h = X_0^k + h'$ for some homogeneous $h' \in T_k$. Therefore we get

$$\sum_{i \geq 0} \text{HF}(T/\tilde{I}, i) = k \prod_{i=1}^n a_i. \quad (3.2.6)$$

Since t is of degree k , using Theorem 2.1.2 on e-compatibility we obtain

$$\text{Rk } F \geq \frac{1}{k} \left(\sum_{i \geq 0} \text{HF}(T/\tilde{I}, i) \right) = \prod_{i=1}^n a_i. \quad (3.2.7)$$

To obtain an upper bound, we proceed as in Case I(i) for the projective variety

$$\mathbb{W} = V(X_2^{a_2} - X_1^{a_2}, \dots, X_n^{a_n} - X_1^{a_n})$$

and the linear system defined by $\delta = F_{a_1}^\perp$. Hence the proof. □

Example 3.2.2. Consider $I = (X^2 + Y^2 + Z^2, Y^3, Z^3)$ and $F = x^5 - 5x^3y^2 - 5x^3z^2 + 15xy^2z^2$.

By Macaulay2 we verify that $F^\perp = I$. Hence, by Theorem 3.2.1 we obtain $\text{Rk } F = 6$.

3.3 Rank of binomials

In this section, we compute the Waring rank of binomials whose annihilator ideal is a complete intersection. Recently, in [2], the authors have computed the annihilator ideal of binomial forms in three variables. Subsequently, the papers [21, 67] characterize the binomials whose annihilator ideal is a complete intersection. In fact, they have computed $F^\perp$ in this case. This computation shows that such an ideal is a quasi-monomial complete intersection in “most cases”. As a consequence of Theorem 3.2.1, we compute the Waring rank of binomials in these cases. In the remaining cases, we use similar techniques to compute the Waring rank. Hence, we compute the Waring rank of all binomial forms whose annihilator ideal is a complete intersection (CI). Consider a binomial $F = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} (x_0^{b_0} \cdots x_k^{b_k} - x_{k+1}^{b_{k+1}} \cdots x_n^{b_n})$. A necessary condition (see [21, 67] for $F^\perp$ to be a CI is $k = 1$. Therefore we consider the binomial of type $F = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} (x_0^{b_0} - x_1^{b_1} \cdots x_n^{b_n})$. Rearranging the variables if required, we can write $F = F_1 F_2$ where $F_1 = x_0^{a_0} x_1^{a_1} \cdots x_k^{a_k} (x_0^{b_0} - x_1^{b_1} \cdots x_k^{b_k})$ is a binomial with $b_i \neq 0$ for $1 \leq i \leq k$, $b_{k+1} = \cdots = b_n = 0$, and $F_2 = x_{k+1}^{a_{k+1}} \cdots x_n^{a_n}$ for some $1 \leq k \leq n$. Moreover, by [22, Proposition 3.1] we have

$$F^\perp = (F_1^\perp)_{S_1} + (F_2^\perp)_{S_2}$$

where $S_1 = \mathbb{C}[x_0, \dots, x_k]$ and $S_2 = \mathbb{C}[x_{k+1}, \dots, x_n]$. Note that $(F_2^\perp)_{S_2}$ is a monomial complete intersection. Hence $F^\perp$ is a CI whenever $(F_1^\perp)_{S_1}$ is a CI. If $k = 1$, then F_1 is a binary binomial and $(F_1^\perp)_{S_1}$ is always a complete intersection, but may not be a quasi-monomial complete intersection. We use a description of annihilator ideal given by K. Shibata in [67]. Using that, we reprove a result [8, Theorem 3.1] of Brustenga i Moncusí and Masuti and obtain the Waring rank of binary binomials. If $k \geq 2$, then by [21, 67] we have a characterization for $(F_1)^\perp$ to be a CI. In fact, it is a quasi-monomial complete intersection. Hence we use Theorem 3.2.1 to obtain the rank.

We first present the case $k = 1$. To do so, we recall a result by K. Shibata [67] which computes the annihilator ideal of a binary binomial.

Theorem 3.3.1 ([67], Lemma 2.11). *Let $S = \mathbb{C}[x_0, x_1]$, and $T = \mathbb{C}[X_0, X_1]$ be the standard graded polynomial rings. Let*

$$F = x_0^{a_0} x_1^{a_1} (x_0^{b_0} - x_1^{b_0})$$

where $b_0, a_0, a_1 \in \mathbb{N}$ with $b_0 \geq 1, 0 \leq a_0 \leq a_1$. Let

$$v = \min\{i \mid a_0 + 1 \leq i b_0\}, \quad w = \min\{i \mid a_1 + 1 \leq i b_0\}.$$

Then

$$F^\perp = \begin{cases} \left(X_0^{a_0+b_0+1}, M = \sum_{i=0}^v X_0^{ib_0} X_1^{a_1+1-ib_0} \right) & \text{if } v < w; \\ \left(P = \sum_{i=0}^v X_0^{ib_0} X_1^{(v-i)b_0}, Q = \sum_{i=0}^{v-1} X_0^{a_0+1-(v-1-i)b_0} X_1^{a_1+1-ib_0} \right) & \text{if } v = w. \end{cases}$$

It is easy to see that when the polynomials M, P, Q are square-free. Indeed,

$$M(X_0^{b_0} - X_1^{b_0}) = X_0^{(v+1)b_0} X_1^{a_1+1-vb_0} - X_1^{a_1+1+b_0}.$$

Whenever $a_1 + 1 - vb_0 \geq 2$, we obtain M has a multiple factor. Similarly,

$$P(X_0^{b_0} - X_1^{b_0}) = X_0^{(v+1)b_0} - X_1^{(v+1)b_0}.$$

Hence P is always square-free. Note that

$$Q(X_0^{b_0} - X_1^{b_0}) = X_0^{a_0+1+b_0} X_1^{a_1+1-(v-1)b_0} - X_0^{a_0+1-(v-1)b_0} X_1^{a_1+1+b_0}.$$

Hence Q has a multiple factor if $a_0 + 1 - (v-1)b_0 \geq 2$ or $a_1 + 1 - (v-1)b_0 \geq 2$, otherwise it is square free. By Sylvester's algorithm [69, 70], the result below is a direct consequence.

Theorem 3.3.2. *Let $F = x_0^{a_0} x_1^{a_1} (x_0^{b_0} - x_1^{b_0})$ be a binary binomial and notations be as in Theorem 3.3.1. Then we have the following:*

$v < w$	$a_1 + 1 < a_0 + b_0 + 1$ and $a_1 + 1 - vb_0 \geq 2$	$\text{Rk } F = a_0 + b_0 + 1$
	Otherwise	$\text{Rk } F = a_1 + 1.$
$v = w$	$vb_0 \geq a_0 + a_1 + 2 - (v-1)b_0, a_1 + 1 - (v-1)b_0 \leq 1$	$\text{Rk } F = a_0 + a_1 + 2 - (v-1)b_0$
	Otherwise	$\text{Rk } F = vb_0.$

Note that the above result is equivalent to [8, Theorem 3.1]. Rewriting the statement in Theorem 3.3.2, we get $F = x_0^{a_0+b_0} x_1^{a_1} - x_0^{a_0} x_1^{a_1+b_0}$. In terms of notation as in [8, Theorem 3.1], we get $r = a_0, \alpha = b_0, s = a_1$. We claim that under this identification, both results yield the same value of the Waring rank.

Proof of the claim. Consider the Euclidean division of a_0 by b_0 : $a_0 = qb_0 + j$, $0 \leq j < b_0$. Hence $v = q+1$ and $vb_0 = (q+1)b_0 = a_0 + b_0 - j$. Therefore $a_1 + 1 - vb_0 = a_1 + 1 - (a_0 + b_0 - j) = j - \delta + 1$, where $\delta = r + \alpha - s = a_0 + b_0 - a_1$.

Case I. We first consider the case $v < w$. Since $w = \lceil (a_1 + 1)/b_0 \rceil$, we have

$$v < w \iff a_1 + 1 > vb_0 \iff j - \delta + 1 > 0 \iff j \geq \delta.$$

Case I.A. Suppose $a_1 + 1 < a_0 + b_0 + 1$ and $a_1 + 1 - vb_0 \geq 2$. This means $a_1 < a_0 + b_0 \iff \delta > 0$, and $j - \delta + 1 \geq 2 \iff j > \delta$. Thus, under the case $\delta > 0, j > \delta$ we have

$$\text{Rk}(F) = a_0 + b_0 + 1 = r + \alpha + 1.$$

Case I.B. In all remaining cases with $v < w$, we have

$$\text{Rk}(F) = a_1 + 1 = s + 1.$$

This matches exactly the cases of [8, Theorem 3.1], where either $\delta \leq 0$, or $\delta > 0$ and $j = \delta$.

Case II. Assume $v = w$. This implies

$$v = w \iff a_1 + 1 \leq vb_0 \iff j - \delta + 1 \leq 0 \iff j < \delta.$$

Case II. A. Assume $vb_0 \geq a_0 + a_1 + 2 - (v - 1)b_0$ and $a_1 + 1 - (v - 1)b_0 \leq 1$. Since $v = q + 1$ the second inequality becomes

$$a_1 + 1 - qb_0 \leq 1.$$

Using $a_1 = a_0 + b_0 - \delta = qb_0 + j + b_0 - \delta$, we obtain

$$j + b_0 - \delta \leq 0.$$

Since $\delta \leq b_0$, we get $j = 0$. Hence $\delta = b_0$. This implies $a_0 = a_1$. Note that $j = 0$ implies $a_0 = qb_0$. Therefore $vb_0 = a_0 + b_0$ and $(v - 1)b_0 = a_0$. Hence, the inequality becomes

$$vb_0 = a_0 + b_0 \geq a_0 + a_1 + 2 - (v - 1)b_0 = a_0 + a_1 + 2 - a_0 = a_0 + 2.$$

This implies $b_0 \geq 2$. Hence this case is the same as $j = 0, r = s, \alpha > 1$, and the Waring rank

$$\text{Rk}(F) = a_0 + a_1 + 2 - (v - 1)b_0 = a_1 + 2 = s + 2.$$

Case II. B. In all remaining cases of $v = w$, the Waring rank is given by

$$\text{Rk}(F) = vb_0 = a_0 + b_0 - j = r + \alpha - j.$$

Hence, the proof of the claim. □

Lemma 3.3.3. *If $\mathbb{V} = V(G, X_3^{a_3+1} - X_2^{a_3+1}, \dots, X_n^{a_n+1} - X_2^{a_n+1}) \subset \mathbb{P}^n, n \geq 2$ is a variety where $G \in \mathbb{C}[X_0, X_1]$ is a homogeneous form and square-free, then the singular locus of $\mathbb{V}$ is a finite set, and is given by*

$$\begin{aligned} \text{Sing}(\mathbb{V}) = & \{[0 : 0 : 1 : \omega_3 : \dots : \omega_n] \mid \omega_i^{a_i+1} = 1 \text{ for all } 2 \leq i \leq n\} \\ & \cup \{[u_0 : u_1 : 0 : 0 : \dots : 0 \mid G(u_0, u_1) = 0\}. \end{aligned}$$

Proof. Note that the Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{\partial G}{\partial X_0} & \frac{\partial G}{\partial X_1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -(a_3+1)X_2^{a_3} & (a_3+1)X_3^{a_3} & 0 & \dots & 0 \\ 0 & 0 & -(a_4+1)X_2^{a_4} & 0 & (a_4+1)X_4^{a_4} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -(a_n+1)X_2^{a_n} & 0 & 0 & \dots & (a_n+1)X_n^{a_n} \end{pmatrix}_{(n-1) \times (n+1)}.$$

If $P = [p_0 : \dots : p_n] \in \text{Sing}(\mathbb{V})$, then $\text{rank } J(P) \leq n-2$. We divide the proof in to two cases.

Case I. We assume $X_2(P) = 0$. Since $P \in \mathbb{V}$ we get $p_2 = \dots = p_n = 0$. Hence the set of points $\{[u_0 : u_1 : 0 : 0 : \dots : 0 \mid G(u_0, u_1) = 0\} \subset \text{Sing}(\mathbb{V})$.

Case II. Assume that $X_2 \neq 0$. All the $(n-2) \times (n-2)$ sub-matrices of $J(P)$ does not have the full rank. This implies $\frac{\partial G}{\partial X_0}(p_0, p_1) = \frac{\partial G}{\partial X_1}(p_0, p_1) = 0$. Since G is square-free, this is not possible unless $p_0 = p_1 = 0$. Hence the proof is complete. $\square$

Lemma 3.3.4. *Let $F^\perp = (P = \sum_{i=0}^v X_0^{ib_0} X_1^{(v-i)b_0}, Q = \sum_{i=0}^{v-1} X_0^{a_0+1-(v-1-i)b_0} X_1^{a_1+1-ib_0}, X_2^{a_2}, \dots, X_n^{a_n})$.*

1. *If $I = (X_2)$, then $F^\perp : I = (P, Q, X_2^{a_2-1}, X_3^{a_3}, \dots, X_n^{a_n})$.*
2. *If $I = (X_1)$, then $F^\perp : I = (P, Q/X_1, X_2^{a_2}, \dots, X_n^{a_n})$.*
3. *If $I = (X_0, X_1)$, then $F^\perp : I = (P, Q, X_2^{a_2}, \dots, X_n^{a_n})$.*

Proof. Refer to Lemma [B.4](#) for a proof. $\square$

We now present the computation of the Waring rank of binomials obtained as the multiplication of a binary binomial and a monomial in different sets of variables.

Theorem 3.3.5. Let $F = F_1 F_2$ where $F_1 = x_0^{a_0} x_1^{a_1} (x_0^{b_0} - x_1^{b_0})$, and $F_2 = x_2^{a_2} \cdots x_n^{a_n}$ such that $a_0 \leq a_1$, and $a_2 \leq \cdots \leq a_n$. Let $\lambda = \min\{a_2 + 1, a_0 + b_0 + 1\}$. Using the notations as in Theorem 3.3.1 if $v < w$, then

$$\text{Rk } F = \begin{cases} \frac{(a_0 + b_0 + 1)}{\lambda} \prod_{i=1}^n (a_i + 1) & \text{if } \lambda \leq a_1 + 1 \text{ or } (a_1 + 1 < \lambda \text{ and } a_1 + 1 - vb_0 \leq 1) \\ (a_0 + b_0 + 1) \prod_{i=2}^n (a_i + 1) & \text{otherwise.} \end{cases}$$

If $v = w$, then we have the following cases:

$$\text{Rk } F = \begin{cases} \deg P \deg Q \left(\prod_{i=3}^n (a_i + 1) \right) & \text{if one of the following holds:} \\ & (i) \ a_2 + 1 \leq \min\{\deg P, \deg Q\} \\ & (ii) \ \deg P \leq a_2 + 1 \leq \deg Q \\ & (iii) \ \deg Q \leq a_2 + 1 \leq \deg P \text{ and } Q \text{ is square-free} \\ \text{Rk } F_1 \left(\prod_{i=2}^n (a_i + 1) \right) & \text{otherwise.} \end{cases}$$

Proof. We divide the proof into two main cases.

Case A: We first prove the case $v < w$. By Theorem 3.3.1 we have

$$F^\perp = \left(X_0^{a_0+b_0+1}, M = \sum_{i=0}^v X_0^{ib_0} X_1^{a_1+1-ib_0}, X_2^{a_2+1}, \dots, X_n^{a_n+1} \right)$$

a quasi-monomial complete intersection. Moreover, M is not square-free if $a_1 + 1 - vb_0 \geq 2$.

Let $\lambda = \min\{a_2 + 1, a_0 + b_0 + 1\}$. Hence by Theorem 3.2.1, we obtain

$$\text{Rk } F = \begin{cases} \frac{(a_0 + b_0 + 1)}{\lambda} \prod_{i=1}^n (a_i + 1) & \text{if } \lambda \leq a_1 + 1 \text{ or } (a_1 + 1 < \lambda \text{ and } a_1 + 1 - vb_0 \leq 1) \\ (a_0 + b_0 + 1) \prod_{i=2}^n (a_i + 1) & \text{otherwise.} \end{cases}$$

Case B: We now prove the case $v = w$. Proof of this case has two main sub parts.

Case I. Let $a_2 + 1 \leq \min\{\deg P, \deg Q\}$ or $\deg P \leq a_2 + 1 \leq \deg Q$ or $(\deg Q \leq a_2 + 1 \leq \deg P$ and Q is square-free). We first obtain a lower bound on the $\text{Rk } F$. Consider $I = (X_2)$, and $t = X_2$. By Proposition 3.1.1 we obtain $F^\perp : (X_2)$. Then by the e-computability Theorem 2.1.2 we get a lower bound

$$\text{Rk } F \geq \deg P \deg Q \left(\prod_{i=3}^n (a_i + 1) \right).$$

To obtain an upper bound on the $\text{Rk } F$, we consider the variety

$$\mathbb{V} = \begin{cases} V(X_3^{a_3+1} - X_2^{a_3+1}, \dots, X_n^{a_n+1} - X_2^{a_n+1}, P) & \text{if } a_2 + 1 \leq \min\{\deg P, \deg Q\} \text{ or;} \\ & \deg P \leq a_2 + 1 \leq \deg Q \\ V(X_3^{a_3+1} - X_2^{a_3+1}, \dots, X_n^{a_n+1} - X_2^{a_n+1}, Q) & \deg Q \leq a_2 + 1 \leq \deg P \text{ and } Q \text{ is square-free,} \end{cases}$$

and a linear system defined by the vector space

$$\Delta = \begin{cases} F_{\deg Q}^\perp & \text{if } a_2 + 1 \leq \min\{\deg P, \deg Q\} \\ F_{\deg P}^\perp & \text{if } \deg Q \leq a_2 + 1 \leq \deg P. \end{cases}$$

By Lemma 3.3.3 we get $\mathbb{V}$ has finitely many singular points. Moreover, we *claim* that the linear system defined by Δ is base point free on $\mathbb{V}$. i.e., $\text{BS}(\Delta) = \emptyset$.

Proof of the claim. Assume by contradiction that $q = [q_0 : q_1 : \dots : q_n] \in \text{BS}(\Delta)$. Hence $P(q) = Q(q) = 0$ because $q \in \mathbb{V} \cap V(\Delta)$. This implies $q_0 = q_1 = 0$. Similarly, note that $X_2^{\deg Q} \in F_{\deg Q}^\perp$ if $a_2 + 1 \leq \deg Q$ and $X_2^{\deg P} \in F_{\deg P}^\perp$ if $a_2 + 1 \leq \deg P$. Hence $q_2 = 0$ in both the cases. Further, $q \in \mathbb{V}$ implies $q_2 = \dots = q_n = 0$. Hence the claim. $\square$

Hence by Remark 3.1.5 we can apply Bertini's Theorem 3.1.4 on the variety $\mathbb{V} \setminus \text{Sing}(\mathbb{V})$ and the linear system defined by Δ on $\mathbb{V} \setminus \text{Sing}(\mathbb{V})$. Hence we obtain an upper bound

$$\text{Rk } F \leq \deg P \deg Q \left(\prod_{i=3}^n (a_i + 1) \right).$$

Case II. If conditions of Case I are not satisfied, then either $\deg Q \leq \min\{a_2 + 1, \deg P\}$ and Q is not square-free, or $\deg P \leq \deg Q < a_2 + 1$. We compute $\text{Rk } F$ by dividing the cases into three subparts. First we obtain a lower bound for $\text{Rk } F$.

Case II(a) Assume that $\deg Q \leq \min\{a_2 + 1, \deg P\}$ and Q is not square free. This gives $a_1 + 1 - (v - 1)b_0 \geq 2$. Therefore $Q = X_1^2 Q'$ for some $Q' \in T$. Consider $I = (X_1)$, and $t = X_1$. Then by Lemma 3.3.4 we have

$$\begin{aligned} F^\perp : I + (t) &= (P, X_1 Q', X_2^{a_2+1}, \dots, X_n^{a_n+1}, X_1) \\ &= (P, X_2^{a_2+1}, \dots, X_n^{a_n+1}, X_1). \end{aligned}$$

Hence by computing the Hilbert function we get

$$\sum_{i \geq 0} \text{HF}(T/(F^\perp : I + (t)), i) = \deg P \prod_{i=2}^n (a_i + 1).$$

Case II(b) If $\deg P \leq \deg Q < a_2 + 1$, then consider $I = (X_0, X_1)$, and $t = X_0 + X_1$. By Lemma 3.3.4 we have

$$F^\perp : I = (P, Q, X_2^{a_2+1}, \dots, X_n^{a_n+1}).$$

Hence we obtain

$$F^\perp : I + (t) = (P, Q, X_2^{a_2+1}, \dots, X_n^{a_n+1}, X_0 + X_1) \quad (3.3.1)$$

Since $\deg P \leq \deg Q$ we obtain

$$\sum_{i \geq 0} \text{HF}(T/(F^\perp : I + (t)), i) = \deg P \prod_{i=2}^n (a_i + 1).$$

In both the Cases II(a) and II(b), by Theorem 2.1.2 on e-computability, we obtain

$$\text{Rk } F \geq \deg P \prod_{i=2}^n (a_i + 1). \quad (3.3.2)$$

We now obtain an upper bound for $\text{Rk } F$ for the Cases II(a) and II(b). Let $\mathbb{V} = V(X_3^{a_3+1} - X_2^{a_3+1}, \dots, X_n^{a_n+1} - X_2^{a_n+1}, P) \subset \mathbb{P}^n$ and a linear system defined by the vector space $\Delta = F_{a_2+1}^\perp$. We claim that Δ is base point free on $\mathbb{V}$.

Proof of the claim. On the contrary, assume that $q = [q_0 : \dots : q_n] \in \text{BS}(\Delta)$. Note that $X_0^{a_2+1-\deg Q} Q, X_1^{a_2+1-\deg Q} Q \in \Delta$. Hence $q_0 = q_1 = 0$ or $Q(q) = 0$. Moreover $q \in \mathbb{V}$, hence $P(q) = 0$. This implies $q_0 = q_1 = 0$. Further, we proceed similarly to Case I to obtain the claim. $\square$

By Lemma 3.3.3 $\mathbb{V}$ has finitely many singular points. By Remark 3.1.5 we use Bertini's Theorem 3.1.4 on the variety $\mathbb{V} \setminus \text{Sing}(\mathbb{V})$ and the linear system defined by Δ on $\mathbb{V} \setminus \text{Sing}(\mathbb{V})$ to obtain an upper bound

$$\text{Rk } F \leq \deg P \left(\prod_{i=2}^n (a_i + 1) \right). \quad (3.3.3)$$

Moreover in the Cases II(a) and II(b), by Theorem 3.3.2 we have $\text{Rk } F_1 = \deg P$. Hence by Equations (3.3.2) and (3.3.3) we have $\text{Rk } F = \text{Rk } F_1 \left(\prod_{i=2}^n (a_i + 1) \right)$.

Case II(c). Assume $\deg Q \leq \deg P < a_2 + 1$ and Q is square-free. We obtain a lower bound on the $\text{Rk } F$ using e-computability. Let $I = (X_0, X_1)$, and $t = X_0 + X_1$. By Lemma 3.3.4 we obtain $F^\perp : I + (t)$. Here $\deg Q \leq \deg P$. We perform computations similar to the Case II(b) to obtain

$$\sum_{i \geq 0} \text{HF}(T/(F^\perp : I + (t)), i) = \deg Q \prod_{i=2}^n (a_i + 1).$$

Thus, by the e-computability Theorem [2.1.2](#) we get a lower bound

$$\text{Rk } F \geq \deg Q \prod_{i=2}^n (a_i + 1). \quad (3.3.4)$$

To obtain an upper bound on $\text{Rk } F$, we consider the variety

$$\mathbb{V} = V(X_3^{a_3+1} - X_2^{a_3+1}, \dots, X_n^{a_n+1} - X_2^{a_n+1}, Q)$$

and a linear system defined by the vector space $\Delta = F_{a_2+1}^\perp$ on $\mathbb{V}$. By using arguments similar to the Case II(b), we obtain an upper bound

$$\text{Rk } F \leq \deg Q \left(\prod_{i=2}^n (a_i + 1) \right). \quad (3.3.5)$$

Again by Theorem [3.3.2](#) in this case we have $\text{Rk } F_1 = \deg Q$. Hence by Equations [\(3.3.4\)](#) and [\(3.3.5\)](#) we have proved $\text{Rk } F = \text{Rk } F_1 \left(\prod_{i=2}^n (a_i + 1) \right)$. □

We now present the case $k \geq 2$. By the results of [\[2, Theorem 4.1\]](#), [\[21, Theorem 3.3\]](#), and [\[67, Theorem 1.3\]](#) we know that if $F^\perp$ is a complete intersection, then it is a quasi-monomial complete intersection. Hence $\text{Rk } F$ is *almost* immediate from Theorem [3.2.1](#). First, we state the known results describing $F^\perp$.

Theorem 3.3.6 (Theorem 4.1 [\[2\]](#), Theorem 3.3 [\[21\]](#), Theorem 1.3 [\[67\]](#)). *Let*

$$F = x_0^{a_0} \cdots x_k^{a_k} (x_0^{b_0} - x_1^{b_1} \cdots x_k^{b_k})$$

be a binomial form with all $b_i \neq 0$ such that $\sum_{i=1}^n b_i = b_0$ and $k \geq 2$, and $a_i \geq 0$. If there exists an integer $1 \leq i \leq k$ with $a_i < qb_i$ where $q = \lfloor \frac{a_0 + 1}{b_0} \rfloor$, then

$$F^\perp = (X_0^{a_0+1} + G, X_1^{a_1+b_1+1}, \dots, X_k^{a_k+b_k+1})$$

where $G = \sum_{j=1}^m (X_1^{b_1} \cdots X_k^{b_k})^j X_0^{a_0+1-jb_0}$ with $m = \min\{\lfloor \frac{a_i}{b_i} \rfloor \mid 1 \leq i \leq k\}$.

We now present our results on the Waring rank of binomials whose annihilator ideal is a quasi-monomial complete intersection as in Theorem [3.3.6](#).

Theorem 3.3.7. *Let $F = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} (x_0^{b_0} - x_1^{b_1} \cdots x_n^{b_n})$. Write $F = F_1 F_2$ where $F_1 = x_0^{a_0} x_1^{a_1} \cdots x_k^{a_k} (x_0^{b_0} - x_1^{b_1} \cdots x_k^{b_k})$ a binomial with $b_i \neq 0$ for $1 \leq i \leq k$, $b_{k+1} = \cdots = b_n = 0$,*

and $F_2 = x_{k+1}^{a_{k+1}} \cdots x_n^{a_n}$ for some $1 \leq k \leq n$, and $a_{k+1} \leq \cdots \leq a_n$.

If $2 \leq k \leq n$, and F_1 satisfies conditions as in Theorem [3.3.6](#) then

$$rk(F) = \begin{cases} \frac{(a_0 + 1)}{\lambda} \prod_{i=1}^k (a_i + b_i + 1) \prod_{i=m+1}^n (a_i + 1) & \text{if } \lambda \leq a_0 + 1 \\ \prod_{i=1}^k (a_i + b_i + 1) \prod_{i=k+1}^n (a_i + 1) & \text{otherwise} \end{cases}$$

where $\lambda = \min(\{a_i + b_i + 1 \mid 1 \leq i \leq k\} \cup \{a_{k+1} + 1\})$.

Proof. By Theorem [3.3.6](#) we have

$$F^\perp = (X_0^{a_0+1} + G, X_1^{a_1+b_1+1}, \dots, X_k^{a_k+b_k+1}, X_{k+1}^{a_{k+1}+1}, \dots, X_n^{a_n+1})$$

where $G = \sum_{j=1}^m (X_1^{b_1} \cdots X_k^{b_k})^j X_0^{a_0+1-jb_0}$ with $m = \min\{\lfloor \frac{a_i}{b_i} \rfloor \mid 1 \leq i \leq k\}$. Hence $F^\perp$ is a quasi-monomial complete intersection. We *claim* that G is not square-free. That is, we prove that $a_0 + 1 - mb_0 \geq 2$. On the contrary, assume that $a_0 + 1 - mb_0 \leq 1$. This means

$$q = \lfloor \frac{a_0 + 1}{b_0} \rfloor \leq \lfloor \frac{mb_0 + 1}{b_0} \rfloor \leq \lfloor m + \frac{1}{b_0} \rfloor = m. \quad (3.3.6)$$

The last equality follows from the fact that $b_0 = \sum_{i=1}^k b_i \geq k \geq 2$. Moreover, by the definition of m we have $mb_j \leq a_j$ for all $1 \leq j \leq k$. By hypothesis there exists i such that $a_i < qb_i$ for some $1 \leq i \leq k$. Therefore, for the same i , we have $mb_i \leq a_i < qb_i$. Since $b_i > 0$ we get

$$m < q. \quad (3.3.7)$$

Hence, by Equations [\(3.3.6\)](#) and [\(3.3.7\)](#) we have a contradiction. Therefore, we can write $G = X_0^{a_0+1-mb_0}g$ for some $g \in T$. Thus $X_0^{a_0+1} + G$ is not square-free. Hence, by Theorem [3.2.1](#) immediately, we obtain the result. □

We apply the Theorem [3.3.7](#) in a specific case in the example below:

Example 3.3.8. Let $F = x_0^2 x_1^3 x_2^3 x_3^4 (x_0^6 - x_1^4 x_2 x_3) \in \mathbb{C}[x_0, x_1, x_2, x_3]$. By Theorem [3.3.7](#), we obtain the Waring rank $\text{Rk } F = 240$.

Chapter 4

Symmetric decomposition of the Hilbert function of an ideal

Throughout this chapter, let $(R, \mathcal{M})$ be a local ring over a field k where $k = R/\mathcal{M}$ and J an ideal in R such that $A = R/J$ is an Artinian Gorenstein (AG) k -algebra. This chapter is devoted to understand the structure of the Hilbert function of I , that is, the Hilbert function of its associated graded ring $G(I) := \bigoplus_{n \geq 0} I^n/I^{n+1}$ for any ideal I in A . We prove the existence of symmetric decomposition of the Hilbert function of an ideal.

In Section 4.1 we prove some preliminary results and set up required background. In Section 4.2, we introduce a symmetric decomposition of the Hilbert function of an arbitrary ideal in an AG local k -algebra. In Section 4.3, we propose some problems and conjectures on possible I -Gorenstein sequences in a codimension two AG k -algebra. In Section 4.4, we give a complete list of 2-admissible sequences of length at most 3 and with $h_0 = 2$ that are I -Gorenstein sequences. These sequences support our conjecture. To conclude that certain 2-admissible sequences are not I -Gorenstein, we have deeply extracted information coming from a possible symmetric decomposition of this sequence. In order to show that a certain 2-admissible sequence H is an I -Gorenstein sequence we have constructed an explicit AG k -algebra A and an ideal I in A , such that $G(I)$ has the Hilbert function H .

4.1 Preliminary results

In this section, we prove some properties of AG k -algebras.

Lemma 4.1.1. *Let $A = R/J$ be an AG k -algebra of socle degree s . Then there exists a k -linear map $\varphi : A \rightarrow k$ which is non-zero on the socle of A . The k -bilinear map induced by this linear map $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$, namely $\langle a, b \rangle = \varphi(ab)$, is non-degenerate.*

Proof. By Proposition C.1 corresponding to $A = R/J$, there exists a k -linear map $\psi : R \rightarrow k$ such that $\psi|_{\mathcal{M}^{s+1}} = 0, \psi|_{\mathcal{M}^s} \neq 0$. Define $\varphi : A \rightarrow k$ as $\varphi(a + J) = \psi(a)$. First we show φ is well-defined. By Proposition C.1, we have $\psi(Rx) = 0$ for all $x \in J$. Hence, in particular $\psi(x) = 0$ for all $x \in J$. Thus φ is well defined. Moreover, since $\text{Soc } A = \mathcal{M}^s + J/J$, $\varphi(\mathcal{M}^s + J/J) = \psi(\mathcal{M}^s) \neq 0$. Hence φ is non-zero on the socle of A .

Next, we prove that $\langle \cdot, \cdot \rangle$ is non-degenerate. Suppose $a + J \in A = R/J$ is such that $\langle a + J, b + J \rangle = 0$, for all $b \in R$. Then, for all $b \in R$,

$$\langle a + J, b + J \rangle = \varphi(ab + J) = \psi(ab) = 0.$$

Hence by the description of J in Proposition C.1 we get that $a \in J$. Hence $a + J = 0$.

Similarly, if $b + J \in A$ is such that $\langle a + J, b + J \rangle = 0$ for all $a + J \in A$, then $b + J = 0$ in A . Therefore, our claim is proved. $\square$

By abusing the notation we denote the bilinear map $\langle \cdot, \cdot \rangle$ in Proposition 4.1.1 as φ . With the notation as in C.2 we recall the definition of perp and annihilator of an ideal I in A induced by this bilinear map φ :

$$\text{Ann}_A(I) = \{a \in A \mid \varphi(at) = 0, \text{ for all } t \in I\}, \text{ and } I^\perp = \{a \in A \mid \varphi(ta) = 0, \text{ for all } t \in I\}$$

Hence, $\text{Ann}_A(I) = I^\perp$.

Proposition 4.1.2. *Let $A = R/J$ be an AG k -algebra, and $Q \subseteq S \subseteq A$ be ideals in A . Then $(S/Q)^\vee \cong Q^\perp/S^\perp$ where $V^\vee = \text{Hom}(V, k)$.*

Proof. Let $\varphi : A \rightarrow k$ be a k -linear map corresponding to A obtained from Lemma 4.1.1. We have a natural k -linear map $\psi : Q^\perp \rightarrow (S/Q)^\vee$ given by

$$\psi(a) = \psi_a : S/Q \rightarrow k$$

where $\psi_a(s + Q) = \varphi(sa)$ for $s \in S$. Since $a \in Q^\perp$, ψ_a is well defined. Note that

$$\ker \psi = \{a \in Q^\perp \mid \psi(a) = \psi_a = 0\} = \{a \in Q^\perp \mid \varphi(sa) = 0 \text{ for all } s \in S\} = S^\perp \cap Q^\perp = S^\perp.$$

Hence, we have an injective map $\bar{\psi} : Q^\perp/S^\perp \rightarrow (S/Q)^\vee$. Further,

$$\begin{aligned}
\dim(Q^\perp/S^\perp) &= \dim Q^\perp - \dim S^\perp \\
&= \dim A - \dim Q - (\dim A - \dim S) \quad (\text{by Proposition } \boxed{C.2}) \\
&= \dim S - \dim Q \\
&= \dim(S/Q) \\
&= \dim(S/Q)^\vee.
\end{aligned}$$

Hence $\bar{\psi}$ is an isomorphism. Thus $(S/Q)^\vee \cong Q^\perp/S^\perp$. $\square$

4.2 Symmetric decomposition of the Hilbert function of $G(I)$

Let $(R, \mathcal{M})$ be a local ring over a field k with $k = R/\mathcal{M}$ and J be an ideal in R such that $A = R/J$ is an AG k -algebra. Let I be an ideal in A . We define the *end degree of I* as the least integer u such that $I^u \neq 0$ and $I^{u+1} = 0$. Then we define the following decreasing sequence of ideals of $G(I)$ as follows:

$$G(I) = C(0) \supseteq C(1) \supseteq \cdots \supseteq C(u+1) = 0$$

where $C(a)$ is a graded ideal of $G(I)$ with i^{th} component given by

$$C(a)_i = \frac{(0 : I^{u+1-a-i}) \cap I^i}{(0 : I^{u+1-a-i}) \cap I^{i+1}} \cong \frac{(0 : I^{u+1-a-i}) \cap I^i + I^{i+1}}{I^{i+1}}. \quad (4.2.1)$$

Define $Q(a) := C(a)/C(a+1)$. We note that

$$\begin{aligned}
Q(a)_v &= C(a)_v / C(a+1)_v \\
&= \frac{[(0 : I^{u+1-a-v}) \cap I^v + I^{v+1}]/I^{v+1}}{[(0 : I^{u+1-a-1-v}) \cap I^v + I^{v+1}]/I^{v+1}}.
\end{aligned}$$

For simplification, let $l = u + 1 - a - v$. Then

$$Q(a)_v = \frac{(0 : I^l) \cap I^v + I^{v+1}}{(0 : I^{l-1}) \cap I^v + I^{v+1}}. \quad (4.2.2)$$

Since $(0 : I^{l-1}) \subseteq (0 : I^l)$, we can write

$$(0 : I^l) \cap I^v + I^{v+1} = (0 : I^l) \cap I^v + (0 : I^{l-1}) \cap I^v + I^{v+1}.$$

Therefore

$$Q(a)_v = \frac{(0 : I^l) \cap I^v + (0 : I^{l-1}) \cap I^v + I^{v+1}}{(0 : I^{l-1}) \cap I^v + I^{v+1}}. \quad (4.2.3)$$

Let $M := (0 : I^l) \cap I^v$, $N := (0 : I^{l-1}) \cap I^v + I^{v+1}$. Then $Q(a)_v = (M+N)/N \cong M/(M \cap N)$.

Now

$$\begin{aligned} M \cap N &= ((0 : I^l) \cap I^v) \cap (I^{v+1} + (0 : I^{l-1}) \cap I^v) \\ &= [(0 : I^l) \cap I^v \cap I^{v+1}] + [(0 : I^{l-1}) \cap I^v] \quad (\text{since } (0 : I^l) \cap I^v \supseteq (0 : I^{l-1}) \cap I^v) \\ &= (0 : I^l) \cap I^{v+1} + (0 : I^{l-1}) \cap I^v. \end{aligned} \tag{4.2.4}$$

Hence

$$Q(a)_v \cong \frac{I^v \cap (0 : I^l)}{I^{v+1} \cap (0 : I^l) + I^v \cap (0 : I^{l-1})} \tag{4.2.5}$$

where $l = u + 1 - a - v$.

The following lemma is a key tool for proving our main results.

Lemma 4.2.1. *Let $A = R/J$ be an AG k -algebra, I an ideal of A , and ϕ be a map corresponding to A as in Lemma 4.1.1. Then $\text{Ann}_A(I) = I^\perp = (0 : I)$, and $I = (0 : I)^\perp$.*

Proof. By definition $\text{Ann}_A(I) = \{a \in A \mid \varphi(aI) = 0\}$. Let $a \in \text{Ann}_A(I)$. Then

$$\varphi(aIA) = \varphi(aI) = 0.$$

Since φ is non-degenerate, we get $aI = 0$. Hence $a \in (0 : I)$.

Conversely, if $x \in (0 : I)$, then $\varphi(xI) = \varphi(0) = 0$. Therefore $x \in \text{Ann}_A(I)$. Hence $\text{Ann}_A(I) = (0 : I)$. As $\text{Ann}_A(I) = I^\perp$, we get $I^\perp = (0 : I)$.

Thus $(\text{Ann}_A(I))^\perp = (0 : I)^\perp$. Since $(\text{Ann}_A(I))^\perp = I$ by Proposition C.2 (ii), $I = (0 : I)^\perp$. $\square$

Next, we prove the main technical result that shows that the Hilbert function of $Q(a)$ is symmetric about $(u - a)/2$.

Proposition 4.2.2. *Let $A = R/J$ be an AG k -algebra, I an ideal in A with the end degree u . Then for any $a \geq 0$,*

$$Q(a)_v^\sim := \text{Hom}(Q(a), k)_v \cong Q(a)_{u-a-v}.$$

Proof. Let $l = u + 1 - (a + v)$. By Equation (4.2.5), we have

$$Q(a)_v^\sim \cong \left(\frac{I^v \cap (0 : I^l)}{I^{v+1} \cap (0 : I^l) + I^v \cap (0 : I^{l-1})} \right)^\sim$$

$$\begin{aligned}
&\cong \frac{(I^{v+1} \cap (0 : I^l) + I^v \cap (0 : I^{l-1}))^\perp}{(I^v \cap (0 : I^l))^\perp} && \text{(by Proposition 4.1.2)} \\
&= \frac{(I^{v+1} \cap (0 : I^l))^\perp \cap (I^v \cap (0 : I^{l-1}))^\perp}{(I^v \cap (0 : I^l))^\perp} && \text{(by Proposition C.1 (iv))} \\
&= \frac{((I^{v+1})^\perp + (0 : I^l)^\perp) \cap ((I^v)^\perp + (0 : I^{l-1})^\perp)}{(I^v)^\perp + (0 : I^l)^\perp} && \text{(by Proposition C.2 (iv))} \\
&= \frac{((0 : I^{v+1}) + I^l) \cap ((0 : I^v) + I^{l-1})}{(0 : I^v) + I^l} && \text{(by Lemma 4.2.1)} \\
&= \frac{(0 : I^{v+1}) \cap [(0 : I^v) + I^{l-1}] + I^l}{(0 : I^v) + I^l} && \text{(since } I^l \subseteq (0 : I^v) + I^{l-1}) \\
&= \frac{(0 : I^v) + I^{l-1} \cap (0 : I^{v+1}) + I^l}{(0 : I^v) + I^l} && \text{(since } (0 : I^v) \subseteq (0 : I^{v+1})) \\
&= \frac{I^{l-1} \cap (0 : I^{v+1}) + (0 : I^v) + I^l}{(0 : I^v) + I^l} \\
&\cong \frac{I^{l-1} \cap (0 : I^{v+1})}{[(0 : I^v) + I^l] \cap [I^{l-1} \cap (0 : I^{v+1})]} \\
&= \frac{I^{l-1} \cap (0 : I^{v+1})}{[[(0 : I^v) + I^l] \cap (0 : I^{v+1})] \cap I^{l-1}} \\
&= \frac{I^{l-1} \cap (0 : I^{v+1})}{[(0 : I^v) + I^l \cap (0 : I^{v+1})] \cap I^{l-1}} && \text{(since } (0 : I^v) \subseteq (0 : I^{v+1})) \\
&= \frac{I^{l-1} \cap (0 : I^{v+1})}{(0 : I^v) \cap I^{l-1} + I^l \cap (0 : I^{v+1})} && \text{(since } I^l \cap (0 : I^{v+1}) \subseteq I^{l-1}) \\
&= Q(a)_{l-1}. && \text{(by Equation 4.2.5)} \\
&= Q(a)_{u-a-v} && \text{(since } l = u + 1 - (a + v)).
\end{aligned}$$

□

The following is one of the main results of this chapter:

Theorem 4.2.3. *Let $A = R/J$ be an AG k -algebra, and I an ideal in A with the end degree u . Then*

$$\text{HF}(G(I))_i = \sum_{a=0}^u H(a)_i,$$

where $H(a)$ denotes the Hilbert function of $Q(a)$. Moreover, $H(a)$ is symmetric about $(u-a)/2$.

Proof. We have $G(I) = C(0) \supseteq C(1) \supseteq \cdots \supseteq C(u+1) = 0$. By considering the Hilbert function, we obtain $\text{HF}(G(I))_i = \sum_{a=0}^u H(a)_i$. Further, by Proposition 4.2.2, we have

$$Q(a)_v \cong Q(a)_{u-a-v}.$$

Hence $H(a)_v = H(a)_{u-a-v}$. This proves the result. □

Next, we give an example to illustrate Theorem 4.2.3. For this, we recall a result by C. Blancafort and S. Nollet which characterizes the Hilbert functions of standard graded algebras over an Artinian local ring that need not be a field. We remark that possible Hilbert functions of standard graded algebras over a field are classically known due to Macaulay [50].

Theorem 4.2.4 ([5], Theorem 2.3). *Let R_0 be an Artinian ring, $R = R_0[x_1, \dots, x_b]$, and $H : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Define the functions q and r by the Euclidean division as $H(n) = \binom{n+b-1}{b-1}q(n) + r(n)$ where $0 \leq r(n) < \binom{n+b-1}{b-1}$. Then H is the Hilbert function of R/I for a homogeneous ideal $I \subset R$ if and only if*

(i) $H(0) \leq \ell(R_0)$ where $\ell(R_0)$ denotes the length of R_0 ;

(ii) $H(n+1) \leq \binom{n+b}{b-1}q(n) + r(n)_{n+}^+$ for all $n \geq 0$.

Here for the n^{th} Macaulay representation $r(n)_n = \binom{k(n)}{n} + \dots + \binom{k(1)}{1}$ of $r(n)$,

$$r(n)_{n+}^+ := \binom{k(n)+1}{n+1} + \dots + \binom{k(1)+1}{1+1}.$$

Note that the integer b is the embedding dimension of the ring R . A numerical function H satisfying the equivalent condition above is called a *b-admissible sequence*.

Definition 4.2.5. Let H be a b -admissible sequence. We say H is an *I-Gorenstein sequence* if there exists a Gorenstein ring A and an ideal $I \subset A$ such that the Hilbert function of $G(I)$ is H .

Corollary 4.2.6. *Let $H = (h_0, h_1, \dots, h_u)$ be a b -admissible sequence. If $h_u > h_0$, then H is not an I-Gorenstein sequence.*

Proof. Suppose H is an I -Gorenstein sequence. Notice that by Equation (4.2.5), $Q(a)_u = 0$ for all $a > 0$. Hence by Theorem 4.2.3, $h_u = H(0)_u$. By symmetry of $H(0)$, we get $h_u = H(0)_u = H(0)_0$. Since $H(0)_0 \leq h_0$, we get $h_u \leq h_0$. This is a contradiction. $\square$

Next, we give an example to illustrate Theorem 4.2.3.

Example 4.2.7. Let $A = \frac{k[[t^4, t^5, t^6]]}{(t^{10})}$. We know that $R = k[[t^4, t^5, t^6]]$ is Gorenstein, since the numerical semigroup ring generated by 4, 5, 6 is symmetric (see [7], Theorem 4.4.8). Further, t^{10} is a nonzerodivisor in R . Hence A is an AG k -algebra. Let $I = (t^6)$ be an ideal in A . Then $I^2 = (t^{12})$, and $I^3 = 0$. Hence the end degree of I is 2.

Note that $\ell(A) = 10$, $\ell(A/I) = 6$, $\ell(A/I^2) = 8$. Hence the Hilbert function of $G(I)$ is $(6,2,2)$. Now we compute $C(a)$ to obtain $Q(a)$. We have

$$(0 : I) = (t^4, t^8, t^9, t^{12}, t^{13}, t^{17}),$$

$$(0 : I^2) = (t^4, t^6, t^8, t^9, t^{11}, t^{12}, t^{13}, t^{17}).$$

$\cap$	I	I^2
$(0 : I)$	(t^{12}, t^{17})	I^2
$(0 : I^2)$	I	I^2

Therefore

$$C(1)_0 = \frac{(0 : I^2) \cap A}{(0 : I^2) \cap I} = \frac{(0 : I^2)}{I}$$

$$C(1)_1 = \frac{(0 : I) \cap I}{(0 : I) \cap I^2} = \frac{(t^{12}, t^{17})}{I^2} = 0$$

$$C(1)_2 = 0$$

$$C(2)_0 = \frac{(0 : I) \cap A}{(0 : I) \cap I} = \frac{(0 : I) + I}{I} = \frac{(t^4, t^6, t^8, t^9, t^{11}, t^{12}, t^{13}, t^{17})}{I} = \frac{(0 : I^2)}{I}$$

$$C(2)_1 = C(3)_0 = 0.$$

Hence

$$Q(0)_0 = C(0)_0/C(1)_0 \cong A/(0 : I^2)$$

$$Q(0)_1 = C(0)_1/C(1)_1 \cong I/I^2$$

$$Q(0)_2 = C(0)_2/C(1)_2 \cong I^2$$

$$Q(1)_0 = C(1)_0/C(2)_0 = 0$$

$$Q(1)_1 = Q(1)_2 = 0$$

$$Q(2)_0 = C(2)_0/C(3)_0 \cong \frac{(0 : I^2)}{I}$$

$$Q(2)_1 = Q(2)_2 = 0.$$

Therefore we have a decomposition of the Hilbert function given below:

HF	6	2	2
H(0)	2	2	2
H(1)	0	0	0
H(2)	4	0	0

The Hilbert series computation of $G(I)$ can also be verified in COCOA as follows:

```

Use S:=QQ[x,y,z];
J:=Ideal(y^2-x*z,y^2,x*z,x^3-z^2);
I:=Ideal(z)+J;
PS:=PrimaryHilbertSeries(J,I); PS;

```

Next, we show that, as in the case of symmetric decomposition of $G(\mathfrak{m})$, $Q(0)$ is a Gorenstein quotient of $G(I)$ of socle degree u (see [Theorem 1.5] [38]).

Proposition 4.2.8. *Let $A = R/J$ be an AG k -algebra, and I an ideal in A with the end degree u . Then $G(I)/C(1) \cong Q(0)$ is a graded Gorenstein quotient of $G(I)$ of socle degree u .*

Proof. Consider $\mathfrak{M} = \frac{\mathfrak{m}}{(0 : I^u)} + Q(0)_{i \geq 1}$, the maximal ideal of $Q(0)$. We prove $(0 : \mathfrak{M})$ is one-dimensional. Let $i < u$ and $\bar{a} \in Q(0)_i$ be such that $\bar{a}\mathfrak{M} = 0$. Then $\bar{a} \cdot Q(0)_1 = 0$. Hence by Equation (4.2.2), we obtain

$$a \cdot I \subseteq (0 : I^{u-i-1}) \cap I^{i+1} + I^{i+2} \subseteq (0 : I^{u-i-1}).$$

Hence

$$a \in (0 : I^{u-i-1}) : I = (0 : I^{u-i}).$$

This gives $\bar{a} = 0$. Therefore $(0 : \mathfrak{M})$ has no element of degree $< u$.

Now suppose $\bar{x} \in Q(0)_u$ is such that $\bar{x} \in (0 : \mathfrak{M})$. Then $\bar{x} \in \left(0 : \frac{\mathfrak{m}}{(0 : I^u)}\right)$. Again, by using Equation (4.2.2), we get

$$x\mathfrak{m} \subseteq (0 : I^0) \cap I^u = 0.$$

Therefore $x \in (0 : \mathfrak{m})$. Hence we obtain $(0 : \mathfrak{M})_u = (0 : \mathfrak{m}) \cap I^u$. Since A is a Gorenstein local ring, and an essential extension of $(0 : \mathfrak{m})$, we obtain that $(0 : \mathfrak{M})_u = (0 : \mathfrak{m}) \cap I^u$ is a one-dimensional k -vector space. Hence $Q(0)$ is Gorenstein of socle degree u . $\square$

Next, we provide equivalent criteria for $G(I)$ to be Gorenstein. Note that the equivalence of (i) and (v) in the following result was obtained in [31, Theorem 4.2]. Here we provide an alternative proof of the same with some more equivalent conditions in terms of the vanishing of $Q(a)$ for $a > 0$.

Theorem 4.2.9. *Let $A = R/I$ be an AG k -algebra, and I an ideal in A with the end degree u . Then the following are equivalent:*

- (i) $G(I)$ is Gorenstein of socle degree u ;
- (ii) $C(1) = 0$;
- (iii) $C(i) = 0$ for all $i \geq 1$;
- (iv) $Q(i) = 0$ for all $i \geq 1$;
- (v) $\text{HF}(G(I))$ is symmetric.

Proof. (i) $\iff$ (ii) : Suppose $G(I)$ is Gorenstein of socle degree u . By definition 4.2.1 we have

$$C(1)_i = ((0 : I^{u-i}) \cap I^i + I^{i+1}) / I^{i+1}.$$

We claim that $(0 : I^{u-i}) = I^{i+1}$ for all $0 \leq i \leq u$. Hence $C(1)_i = 0$ for $0 \leq i \leq u$. Indeed, $I^{i+1} \subseteq (0 : I^{u-i})$. Conversely, suppose there exists $\alpha \in (0 : I^{u-i}) \setminus I^{i+1}$. Choose $j < i + 1$ such that $\alpha \in I^j \setminus I^{j+1}$. Consider the ideal $J = (\alpha^*)$ in $G(I)$ where α^* denotes the initial form of α in $G(I)$. Since $G(I)$ is an essential extension of $\text{Soc } G(I)$ and $G(I)$ is Gorenstein of socle degree u , we get $[\text{Soc } G(I)]_u \cap J_u \neq 0$. Therefore $0 \neq \alpha^* x^* \in \text{Soc } G(I) \cap J$ for some $x^* \in G(I)_{u-j}$. Since $x \in I^{u-j} \subseteq I^{u-i}$ and $\alpha \in (0 : I^{u-i})$, we get $\alpha x = 0$ which is a contradiction. Therefore we have proved the claim.

Conversely, $C(1) = 0$ implies that $G(I) = Q(0)$. Hence $G(I)$ is Gorenstein of socle degree u by Proposition 4.2.8.

(ii) $\iff$ (iii) : By definition $C(1) \supseteq C(i)$, for all $i \geq 2$. Hence if $C(1) = 0$, then $C(i) = 0$ for all $i \geq 2$. The converse is clear.

(iii) $\iff$ (iv) : By definition, $Q(i) = C(i)/C(i+1)$. Hence $Q(i) = 0$ for all $i \geq 1$ if $C(i) = 0$ for all $i \geq 1$. Conversely, suppose $Q(i) = 0$, for all $i \geq 1$. Then

$$C(1) = C(2) = \cdots = C(u) = C(u+1) = 0.$$

(iv) $\iff$ (v) : Suppose $Q(i) = 0$ for all $i \geq 1$. Since $\text{HF}(G(I)) = \sum_{i=0}^u \text{HF}(Q(i))$ by Theorem 4.2.3, $\text{HF}(G(I)) = \text{HF}(Q(0))$. By Theorem 4.2.3, $\text{HF}(Q(0))$ is symmetric, and so does the Hilbert function of $G(I)$.

Conversely, suppose that the Hilbert function of $G(I)$ is symmetric. Note that the Hilbert function of $Q(0)$ is also symmetric. Further, $\text{HF}(G(I)) = \sum_{i=0}^u \text{HF}(a)$ by Theorem 4.2.3. We claim that $Q(a) = 0$ for all $a > 0$. Suppose $Q(a) \neq 0$ for some $a > 0$. Let

$$i := \max\{j : Q(a)_j \neq 0 \text{ for some } a > 0\}.$$

Then $Q(a)_j = 0$ for all $j > i$ and $a > 0$. Hence $\text{HF}(G(I))_{u-k} = H(0)_{u-k}$ for all $u - k > i$. Since the Hilbert functions of $Q(0)$ and $G(I)$ are symmetric,

$$\text{HF}(G(I))_k = \text{HF}(G(I))_{u-k} = H(0)_{u-k} = H(0)_k \text{ for all } k < u - i.$$

In particular, for $k = u - a - i < u - i$ we obtain $H(a)_{u-a-i} = 0$ for all $a > 0$. Hence by Proposition 4.2.2, $H(a)_i = H(a)_{u-a-i} = 0$ for all $a > 0$, which is a contradiction. $\square$

The above result helps us to conclude the Gorensteinness of the associated graded ring by computing the Hilbert function alone.

4.3 Some open problems in codimension two

In this section we assume that $A = \frac{k[[x, y]]}{J}$ is an AG k -algebra. In the codimension two case, Iarrobino proved that the symmetric decomposition of the Hilbert function of $G(\mathfrak{m})$ is unique. We show that for any ideals in A such a result is not necessarily true.

Using Theorem 4.2.4 we have written the code “isHFA” which is useful to understand whether a given sequence is b -admissible. This code is written using the similar ideas of Macaulay2 function “isHF” used to verify admissible Hilbert functions of standard graded rings over the field of rationals. But unlike in the code “isHF”, in the code “isHFA” we need to provide an integer b and a numerical sequence h in the input. The output is “true” if h is the Hilbert function of $\frac{R_0[x_1, \dots, x_b]}{I}$ for some ideal I in $R_0[x_1, \dots, x_b]$ where R_0 is Artinian, else is “false”. We use this code to verify that a numerical sequence is b -admissible in this chapter. The code is as follows:

```

loadPackage "LexIdeals"

isHFA = method(TypicalValue=>Boolean)

isHFA(ZZ, List) := (b,hilb) ->
  ( result:=true;
    if not all(hilb,i->instance(i,ZZ)) then result=false
    else( if hilb#1 > b*hilb#0 then result=false
    else (
      leng:=#hilb;
      degr:=1; while result==true and degr < leng-1
      do ((q,r) :=
        quotientRemainder(hilb#(degr), binomial(degr+b-1, b-1));
      if hilb#(degr+1) > q*(binomial(degr+b, b-1))+macaulayBound(r,degr)
      then result=false else degr=degr+1; ));
    );
  result
)

--Example
i1:isHFA(2, {2,2,3,4})
o1:true

```

Let $A = \frac{k[[x, y]]}{J}$ be an AG k -algebra with the maximal ideal $\mathfrak{m}$. Inspired by the results from the symmetric decomposition of the Hilbert function of $G(\mathfrak{m})$ in [36] we ask:

- Question 4.3.1.** *i. Which b -admissible sequences are I -Gorenstein sequences for an ideal I in A ?*
- ii. Does every I -Gorenstein sequence have a unique symmetric decomposition?*

We remark that a characterization of Gorenstein sequence of codimension two is classically known due to F.S. Macaulay [49]¹. Namely, an admissible sequences of the form

$$H(A) = (1, 2, \dots, d, h_d, h_{d+1}, \dots, h_s = 1),$$

¹In [49] Macaulay concluded a proof using his inverse systems and some ideas of C. A. Scott in [65]. The characterization of local Gorenstein sequences was also obtained by Briançon [6], and later Iarrobino in [38, Chapter 2] presented a self-contained proof using symmetric decomposition.

where $d \geq h_d \geq h_{d+1} \geq h_s = 1$, is a Gorenstein sequence if and only if $|h_i - h_{i+1}| \leq 1$ for all i .

Inspired by this result, we guessed that if a b -admissible sequence $(h_0, \dots, h_s)$ is I -Gorenstein for an ideal I in A , then $|h_i - h_{i+1}| \leq h_0$. The following example shows that this need not to be true.

Example 4.3.2. Consider a 3-admissible sequence $h = (3, 7, 7, 3)$. Then h is an I -Gorenstein sequence. Indeed, let $A = \frac{k[[x, y]]}{(x^4 + y^4, x^3y^2)}$ and $I = (x^2, y^2, xy) \subset A$ be an ideal in A . Then A is an AG k -algebra, and $G(I)$ has the Hilbert function h .

Note that here $|h_1 - h_0| = 4 > 3$, and h is not a 2-admissible sequence.

We conjecture the following for a 2-admissible sequence to be I -Gorenstein. We say that a sequence $H = (h_0, h_1, \dots, h_u)$ is *partially strict unimodal* if $h_0 < h_1 < \dots < h_i \geq h_{i+1} \geq \dots \geq h_u$ for some $0 \leq i \leq u$.

Conjecture 4.3.3. Let $A = k[[x, y]]/J$ be an AG k -algebra. Let $H = (h_0, h_1, \dots, h_u)$ be a 2-admissible sequence such that

- (i) H is partially strict unimodal; and
- (ii) $|h_i - h_{i+1}| \leq h_0$ for all i .

Then H is an I -Gorenstein sequence.

A reason for assuming H is partially strict unimodal is that if H is an O-sequence of the form $(1, 2, \dots, h_s)$, then by Macaulay's condition H is partially strict unimodal, which need not be true if H is a b -admissible sequence. In the following we give an example of a 2-admissible sequence, which shows that the assumption (i) in Conjecture 4.3.3 is necessary.

Example 4.3.4. Consider a 2-admissible sequence $H = (2, 2, 3, 2)$. Then H is not partially strict unimodal, but $|h_i - h_{i+1}| \leq h_0 = 2$ for all i . But H is not an I -Gorenstein sequence because H does not admit a symmetric decomposition.

In Section 5, we investigate 2-admissible sequences with $h_0 = 2$ and $u \leq 3$, which supports our conjecture.

Now we investigate Question 4.3.1(ii). In [36] Iarrobino showed that if A is an AG k -algebra of codimension two, then the Hilbert function of A determines the symmetric decomposition. Moreover, each $Q(a)$ is isomorphic, after a shift in grading, to a complete intersection

$k[x, y]/(f_a, g_a)$ [38, Theorem 2.2]. The example below shows that a 2-admissible I -Gorenstein sequence need not determine the symmetric decomposition.

Example 4.3.5. I -Gorenstein sequence with two different symmetric decompositions: Consider a 2-admissible sequence $H = (4, 4, 1)$. Let $A = Q[[x, y]]/(x^2y^2 - x^3, y^3)$, and $I_1 = (x^2, y^2)$ an ideal in A . Then the Hilbert function of $G(I_1)$ is H . Further, consider the ring $B = Q[[x, y]]/(y^4 - x^5, xy)$, and the ideal $I_2 = (x^3, y^2)$ in B . Then the Hilbert function of $G(I_2)$ is also H . The Hilbert function of $G(I_2)$ is also H . But $G(I_1)$ and $G(I_2)$ have different symmetric decompositions shown in Table 4.1.

HF($G(I_1)$)	4, 4, 1	HF($G(I_2)$)	4, 4, 1
H(0)	1, 2, 1	H(0)	1, 1, 1
H(1)	2, 2	H(1)	3, 3
H(2)	1		

Table 4.1: Hilbert function with two different symmetric decompositions

We conjecture that a 2-admissible I -Gorenstein sequence with $h_0 = 2$ determines the symmetric decomposition.

Conjecture 4.3.6. *Let H be a 2-admissible $G(I)$ -Gorenstein sequence where I is an ideal in an AG k -algebra $A = k[[x, y]]/J$ with $h_0 = 2$. Then the symmetric decomposition of H is unique.*

4.4 2-admissible I -Gorenstein sequences with the end degree at most 3 and $h_0 = 2$

In this section, we give a complete list of 2-admissible I -Gorenstein sequences with $h_0 = 2$ and of end degree at most 3 where I is an ideal in $A = k[[x, y]]/J$. This list shows that the Conjecture 4.3.3 is true if H is an I -admissible sequence with $h_0 = 2$, and end degree at most 3.

Let I be an ideal in an AG k -algebra $A = k[[x, y]]/J$ such that $G(I)$ has the Hilbert function H with $h_0 = 2$. Then $\ell(A/I) = 2$. The following lemma shows that in this case we can assume that $I = (x, y^2)$, up to a change of coordinates.

Lemma 4.4.1. *Let $A = k[[x, y]]/J$ be an AG k -algebra with an embedding dimension two. If I is an ideal in A such that $\ell(A/I) = 2$, then $I = (x, y^2)$ up to a change of coordinates.*

Proof. Suppose $\ell(A/I) = 2$. Let $\mathfrak{m} = (x, y)A$. Then we have a composition series $I \subset \mathfrak{m} \subset A$ with $\mathfrak{m}/I \cong A/\mathfrak{m}$. Hence we obtain $\mathfrak{m}^2 \subseteq I$. Since $\ell(A/I) = 2$ whereas $\ell(A/\mathfrak{m}^2) = 3$, $\mathfrak{m}^2 \subsetneq I$. We choose an element $v \in I \setminus \mathfrak{m}^2$. Let $\frac{\mathfrak{m}}{I} = \langle u \rangle$ as a k -vector space.

Claim: $\mathfrak{m} = (u, v)$.

By Nakayama's lemma it is enough to show that $\bar{u}, \bar{v}$ are k -linearly independent in $\mathfrak{m}/\mathfrak{m}^2$ where $\bar{u}, \bar{v}$ denote the image of u, v in $\mathfrak{m}/\mathfrak{m}^2$. First note that $u, v \in \mathfrak{m} \setminus \mathfrak{m}^2$. Suppose, u, v are linearly dependent in $\mathfrak{m}/\mathfrak{m}^2$. Then we may assume that $\bar{u} = \bar{c}\bar{v}$ for some $c \in A$. Hence $u - cv \in \mathfrak{m}^2 \subsetneq I$. Since $v \in I$, we obtain that $u \in I$. This is a contradiction because $\bar{u}$ is a nonzero element in $\mathfrak{m}/I$. Hence u, v are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$.

Therefore $\ell(A/(v, u^2)) = 2$. Since we have $(v, u^2) \subset I$ and $\ell(A/(v, u^2)) = \ell(A/I) = 2$, we get $I = (v, u^2)$. Therefore by a change of coordinates, we can assume that $I = (x, y^2)$. □

Proposition 4.4.2. *Let $H = (2, h_1, h_2, \dots, h_{u-2}, h_{u-1}, 1)$ be a b -admissible sequence. If H is an I -Gorenstein sequence for an ideal I in an AG k -algebra $A = k[[x, y]]/J$ with $H(1)_0 = 0$, and $H(0) = (1, 1, \dots, 1)$, then $H(1)_1 \leq 1$.*

Proof. By definition $H(a)_j = 0$ whenever $a + j \geq u + 1$. Therefore by Theorem 4.2.3, we have

$$H(1)_{u-1} + H(0)_{u-1} = h_{u-1}.$$

Since $H(1)_0 = 0$, by Proposition 4.2.2 we have $H(1)_{u-1} = H(1)_0 = 0$.

Therefore, the decomposition table looks like the following:

H	$2, h_1, \dots, h_{u-1}, h_u$
$H(0)$	1, 1, $\dots$ 1, 1
$H(1)$	0 - $\dots$ 0
$H(2)$	- - $\dots$
$\vdots$	

By definition $Q(0)_0 = \frac{A}{(0 : I^u)}$. Since $H(0)_0 = 1$, we get

$$\mathfrak{m} = (0 : I^u). \quad (4.4.1)$$

Moreover, by Equation (4.2.5) $Q(1)_0 = \frac{(0 : I^u)}{I \cap (0 : I^u) + (0 : I^{u-1})}$. Hence $H(1)_0 = 0$ implies that

$$(0 : I^u) = I \cap (0 : I^u) + (0 : I^{u-1}). \quad (4.4.2)$$

Substituting Equation (4.4.1) in Equation (4.4.2), we obtain

$$\mathfrak{m} = I + (0 : I^{u-1}). \quad (4.4.3)$$

Further, by Equation (4.2.5) we have

$$Q(0)_1 = \frac{I \cap (0 : I^u)}{I^2 \cap (0 : I^u) + I \cap (0 : I^{u-1})} = \frac{I}{I \cap (0 : I^{u-1})}.$$

Since $\ell(A/I) = 2$, and $H(0)_1 = 1$, we obtain

$$\ell\left(\frac{A}{I \cap (0 : I^{u-1})}\right) = 3. \quad (4.4.4)$$

By Equation (4.4.3) we have

$$\begin{aligned} \frac{\mathfrak{m}}{I} &= \frac{I + (0 : I^{u-1})}{I} \\ &= \frac{(0 : I^{u-1})}{I \cap (0 : I^{u-1})}. \end{aligned}$$

Since $\ell\left(\frac{\mathfrak{m}}{I}\right) = 1$, we obtain $\ell\left(\frac{(0 : I^{u-1})}{I \cap (0 : I^{u-1})}\right) = 1$. By using Equation (4.4.4), we obtain $\ell\left(\frac{A}{(0 : I^{u-1})}\right) = 2$. Hence by Lemma 4.4.1, after a change of coordinates, we can assume that $(0 : I^{u-1}) = (z, w^2)$ where $\mathfrak{m} = (z, w)$ in A .

Since $\ell(A/I) = 2$, by Lemma 4.4.1 we may assume that $I = (x, y^2)$, up to a change of coordinates.

Claim: $\bar{z}, \bar{x}$ are linearly independent in $\frac{\mathfrak{m}}{\mathfrak{m}^2}$, and hence $\mathfrak{m} = (x, z)$.

Suppose not. Then $z - \alpha x \in \mathfrak{m}^2$ for some $\alpha \in A$. Note that as $\ell(\mathfrak{m}/I) = 1$, $\mathfrak{m}^2 \subseteq I = (x, y^2)$. Hence we obtain $z \in I$, which gives that

$$(0 : I^{u-1}) = (z, w^2) \subseteq I.$$

By Equation (4.4.3) we obtain

$$\mathfrak{m} = I + (0 : I^{u-1}) = I,$$

which is a contradiction. This proves the claim.

Therefore $\ell(A/(x, z^2)) = 2$. As $(x, z^2) \subseteq I$ and $\ell(A/I) = 2$, we obtain $I = (x, z^2)$. Since $z \in (0 : I^{u-1})$, we get $xz, z^3 \in I \cap (0 : I^{u-2})$. Therefore the set $\{1, x, z, z^2\}$ spans the k -vector space $\frac{A}{I^2 + I \cap (0 : I^{u-2})}$, and hence

$$\ell\left(\frac{A}{I^2 + I \cap (0 : I^{u-2})}\right) \leq 4. \quad (4.4.5)$$

By Equations (4.2.5) we have

$$Q(1)_1 = \frac{I \cap (0 : I^{u-1})}{I^2 \cap (0 : I^{u-1}) + I \cap (0 : I^{u-2})} = \frac{I \cap (0 : I^{u-1})}{I^2 + I \cap (0 : I^{u-2})}.$$

Using Equation (4.4.5) and (4.4.4) we obtain

$$H(1)_1 = \ell\left(\frac{A}{I^2 + I \cap (0 : I^{u-2})}\right) - \ell\left(\frac{A}{I \cap (0 : I^{u-1})}\right) \leq 4 - 3 = 1.$$

□

Remark 4.4.3. Proposition 4.4.2 is useful in obtaining information about the symmetric decomposition of the Hilbert function of $G(I)$ in certain cases. For example, if $H = (2, 4, 3, 1, 1)$ is an I -Gorenstein sequence, then by Proposition 4.4.2, we conclude that H has the unique symmetric decomposition given below:

H	2, 4, 3, 1, 1
H(0)	1, 1, 1, 1, 1
H(1)	0 1 1 0
H(2)	1 2 1
H(3)	0 0
H(4)	0

We are now ready to describe the complete set of 2-admissible I -Gorenstein sequences for an ideal I in $A = k[[x, y]]/J$ with the end degree at most 3 and $h_0 = 2$. If $H = (2, h_1, \dots, h_u)$ is an I -Gorenstein sequence, then by Corollary 4.2.6 we get $h_u \leq 2$. Below is the list of all 2-admissible sequences up to $u = 3$ with $h_u \leq 2$, where we indicate the I -Gorenstein sequences in bold.

(2,1)	(2,2)	(2,1,1)	(2,2,1)	(2,2,2)	(2,3,1)	(2,3,2)
(2,4,1)	(2,4,2)	(2,1,1,1)	(2,2,1,1)	(2,2,2,1)	(2,2,2,2)	(2,2,3,1)
(2,2,3,2)	(2,3,1,1)	(2,3,2,1)	(2,3,2,2)	(2,3,3,1)	(2,3,3,2)	(2,3,4,1)
(2,3,4,2)	(2,4,1,1)	(2,4,2,1)	(2,4,2,2)	(2,4,3,1)	(2,4,3,2)	(2,4,4,1)
(2,4,4,2)						

Table 4.2: 2-admissible sequences up to $u=3$ with $h_u \leq 2$.

In Table 4.3 we justify that the sequences indicated in bold in Table 4.2 are I -Gorenstein. In fact, for each such sequence H we provide a polynomial F in the divided power ring $k_{DP}[X, Y]$ so that the apolar algebra $A_F := k[[x, y]]/\text{Ann}_{k[[x, y]]}(F)$ is an AG k -algebra (see [39, Appendix A], [23, Appendix A2.4] for basic properties of divided power rings, and [24] for definition of apolar algebra). Then $I = (x, y^2)$ is an ideal in A_F such that $G(I)$ has the Hilbert function H . Moreover, in Table 4.3 we also describe the symmetric decomposition of the Hilbert function of $G(I)$.

Table 4.3: 2-admissible I -Gorenstein sequences with a symmetric decomposition

I-Gorenstein sequence HF	Form F	symmetric decomposition	I-Gorenstein sequence HF	Form F	symmetric decomposition
(2, 1)	$X + Y^2$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2, 1 \\ 1, 1 \\ 1 \end{smallmatrix}$	(2, 2)	XY	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2, 2 \\ 2, 2 \\ 0 \end{smallmatrix}$
(2, 1, 1)	$X^2 + Y^2$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2, 1, 1 \\ 1, 1, 1 \\ 1, 0 \end{smallmatrix}$	(2, 2, 1)	$X^2 + Y^3$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2, 2, 1 \\ 1, 1, 1 \\ 1, 1 \\ 0 \end{smallmatrix}$
(2, 2, 2)	X^2Y	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2, 2, 2 \\ 2, 2, 2 \end{smallmatrix}$	(2, 3, 1)	XY^2	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2, 3, 1 \\ 1, 2, 1 \\ 1, 1 \\ 0 \end{smallmatrix}$

<i>I</i> -Gorenstein sequence HF	Form F	symmetric decomposition	<i>I</i> -Gorenstein sequence HF	Form F	symmetric decomposition
(2, 3, 2)	$X^2Y + Y^4$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,3,2 \\ 2,3,2 \end{smallmatrix}$	(2, 4, 2)	XY^3	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,4,2 \\ 2,4,2 \end{smallmatrix}$
(2, 1, 1, 1)	$X^3 + Y^2$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,1,1,1 \\ 1,1,1,1 \end{smallmatrix}$ $\text{H}(3) \begin{smallmatrix} 1,0 \end{smallmatrix}$	(2, 2, 1, 1)	$X^3 + Y^3$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,2,1,1 \\ 1,1,1,1 \end{smallmatrix}$ $\text{H}(2) \begin{smallmatrix} 1,1 \end{smallmatrix}$
(2, 2, 2, 1)	$X + Y^6$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,2,2,1 \\ 1,1,1,1 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 1,1,1 \end{smallmatrix}$	(2, 2, 2, 2)	X^3Y	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,2,2,2 \\ 2,2,2,2 \end{smallmatrix}$
(2, 3, 1, 1)	$X^3 + Y^4$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,3,1,1 \\ 1,1,1,1 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 0,1,0,0 \end{smallmatrix}$ $\text{H}(2) \begin{smallmatrix} 1,1 \end{smallmatrix}$	(2, 3, 2, 1)	$X^2 + Y^6$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,3,2,1 \\ 1,1,1,1 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 1,2,1 \end{smallmatrix}$
(2, 3, 2, 2)	$X^2 + Y^7$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,3,2,2 \\ 2,2,2,2 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 0,1,0 \end{smallmatrix}$	(2, 3, 3, 1)	X^2Y^2	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,3,3,1 \\ 1,2,2,1 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 1,1,1 \end{smallmatrix}$
(2, 3, 3, 2)	$X^3 + Y^7$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,3,3,2 \\ 2,3,3,2 \end{smallmatrix}$	(2, 4, 2, 1)	$X^2Y + Y^6$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,4,2,1 \\ 1,1,1,1 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 1,3,1 \end{smallmatrix}$
(2, 4, 2, 2)	$X^2Y + Y^7$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,4,2,2 \\ 2,2,2,2 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 0,2,0 \end{smallmatrix}$	(2, 4, 3, 1)	XY^4	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,4,3,1 \\ 1,2,2,1 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 1,2,1 \end{smallmatrix}$
(2, 4, 3, 2)	$Y^7 + X^2Y^2$	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,4,3,2 \\ 2,3,3,2 \end{smallmatrix}$ $\text{H}(1) \begin{smallmatrix} 0,1,0 \end{smallmatrix}$	(2, 4, 4, 2)	XY^5	$\frac{\text{HF}}{\text{H}(0)} \begin{smallmatrix} 2,4,4,2 \\ 2,4,4,2 \end{smallmatrix}$

Next we prove that each of the sequences that are not bold in the Table 4.2 are not *I*-Gorenstein sequences. Since $\ell(A/I) = 2$, by Lemma 4.4.1 we assume $I = (x, y^2)$.

(2,4,1): Suppose (2,4,1) is an *I*-Gorenstein sequence. Then the possible symmetric decomposi-

tions are

H	2, 4, 1	H	2, 4, 1
H(0)	1, 3, 1	H(0)	1, 4, 1
H(1)	1, 1	H(1)	0, 0
H(2)	0	H(2)	1

Since $H(0)_0 = 1$ in both the symmetric decompositions, by definition of $Q(0)_0$ we obtain $(0 : I^2) = \mathfrak{m}$. Hence $\mathfrak{m}I \subseteq (0 : I)$ which gives that $xy, y^3 \in (0 : I)$. Therefore $\ell(A/I^2 + I \cap (0 : I)) \leq 4$. Since by Equation (4.2.5) we have $Q(0)_1 = \frac{I}{I^2 + I \cap (0 : I)}$, we obtain

$$H(0)_1 = \ell(A/I^2 + I \cap (0 : I)) - \ell(A/I) \leq 4 - 2 = 2.$$

This is a contradiction in both the symmetric decompositions. Hence $(2, 4, 1)$ is not an I -Gorenstein sequence.

(2,2,3,1): If $H = (2, 2, 3, 1)$ is an I -Gorenstein sequence, then $\ell(A/I) = 2$, $\ell(A/I^2) = 4$, $\ell(A/I^3) = 7$, and the only possible symmetric decomposition of H is

H	2, 2, 3, 1
H(0)	1, 2, 2, 1
H(1)	1, 0, 1
H(2)	0, 0
H(3)	0

Since $Q(0)_0 = A/(0 : I^3)$ and $H(0)_0 = 1$, we get $(0 : I^3) = \mathfrak{m}$. Hence $\mathfrak{m}I \subseteq (0 : I^2)$. Therefore we obtain

$$xy, y^3 \in I \cap (0 : I^2). \quad (4.4.6)$$

By Equation (4.2.5) we have

$$Q(1)_0 = \frac{I^0 \cap (0 : I^3)}{I \cap (0 : I^3) + I^0 \cap (0 : I^2)} = \frac{(0 : I^3)}{I + (0 : I^2)}.$$

Since $\ell(A/(0 : I^3)) = 1$ and $H(1)_0 = 1$, we get $\ell(A/I + (0 : I^2)) = 2$. Further, $I \subseteq I + (0 : I^2)$ and $\ell(A/I) = 2$ implies that $I + (0 : I^2) = I$, which gives $(0 : I^2) \subseteq I$.

Again by Equation (4.2.5) we have

$$\begin{aligned} Q(0)_1 &= \frac{I \cap (0 : I^3)}{I^2 \cap (0 : I^3) + I \cap (0 : I^2)} \\ &= \frac{I}{(0 : I^2)} \quad (\text{since } (0 : I^2) \subseteq I). \end{aligned}$$

Since $H(0)_1 = 2$ and $\ell(A/I) = 2$, we get $\ell(A/(0 : I^2)) = 4$. Since $I^2 \subseteq (0 : I^2)$ and $\ell(A/I^2) = 4$, we obtain

$$I^2 = (0 : I^2). \quad (4.4.7)$$

Hence by Equation (4.4.6), we get

$$xy, y^3 \in I^2.$$

Note that $I^2 = (x^2, xy^2, y^4) + J$ in $k[[x, y]]/J$. Since $\ell(k[[x, y]]/J) = 8$, we get $\ell(k[[x, y]]/J^*) = 8$ where J^* denotes the ideal of initial forms of J in $k[[x, y]]$. Since $xy \in I^2 = (x^2, xy^2, y^4) + J$ we obtain

$$xy - c_1 x^2 \in J^* \quad (4.4.8)$$

for some $c_1 \in k$. Similarly, $y^3 \in I^2$, implies that

$$y^3 - (d_1 x + d_2 y)x^2 + d_3 xy^2 \in J^* \quad (4.4.9)$$

for some $d_1, d_2, d_3 \in k$. Since $x^4 \in J$, by Equations (4.4.8) and (4.4.9) we obtain that the set $\{1, x, y, x^2, y^2, x^3\}$ spans the k -vector space $k[[x, y]]/J^*$. Hence $\ell(k[[x, y]]/J^*) \leq 6$ which is a contradiction.

(2,2,3,2): Since $(2, 2, 3, 2)$ does not admit a symmetric decomposition, it is not an I -Gorenstein sequence.

(2,3,4,1): If $H = (2, 3, 4, 1)$ is an I -Gorenstein sequence, then the possible symmetric decomposition of H is

H	2, 3, 4, 1
H(0)	1, 3, 3, 1
H(1)	1, 0, 1

Since $Q(0)_0 = \frac{A}{(0 : I^3)}$ and $H(0)_0 = 1$, we get $(0 : I^3) = \mathfrak{m} = (x, y)$. Hence $\mathfrak{m}I \subseteq (0 : I^2)$. Therefore, $xy, y^3 \in I \cap (0 : I^2)$. This gives that $\ell(A/I^2 + I \cap (0 : I^2)) \leq 4$.

Again, by Equation (4.2.5),

$$Q(0)_1 = \frac{I \cap (0 : I^3)}{I^2 \cap (0 : I^3) + I \cap (0 : I^2)} = \frac{I}{I^2 + I \cap (0 : I^2)}.$$

Since $\ell(A/I) = 2$, and $\ell(A/I^2 + I \cap (0 : I^2)) \leq 4$, we obtain $H(0)_1 \leq 2$. This is a contradiction.

(2,3,4,2): Since $(2, 3, 4, 2)$ does not admit a symmetric decomposition, it is not an I -Gorenstein sequence.

(2,4,1,1): Suppose $H = (2, 4, 1, 1)$ is an I -Gorenstein sequence. Then the possible symmetric decomposition of H is

H	2, 4, 1, 1
H(0)	1, 1, 1, 1
H(1)	0, ≤ 1 , 0
H(2)	-, -, 0
H(3)	- 0

Here we conclude $H(1)_1 \leq 1$ by Proposition 4.4.2. This shows that $H(2)_1 \geq 2$ which implies that $H(2)_0 \geq 2$, a contradiction.

(2,4,4,1): If $H = (2, 4, 4, 1)$ is an I -Gorenstein sequence, then the possible symmetric decompositions of H are

H	2, 4, 4, 1	H	2, 4, 4, 1
H(0)	1, 3, 3, 1	H(0)	1, 4, 4, 1
H(1)	1, 1, 1	H(1)	0, 0
H(2)	0, 0	H(2)	0, 0
H(3)	0	H(3)	1

Since $H(0)_1 = 1$ in both cases, by using the arguments similar to the case $(2, 4, 1)$ we get a contradiction.

Chapter 5

Gorensteinness of associated graded rings of filtrations

This chapter focuses on applications of the Hilbert function in the singularity theory. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d with an infinite residue field. Recall that an F_1 -good filtration, $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a descending chain $A = F_0 \supseteq F_1 \supseteq \cdots$ of ideals satisfying $F_i F_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{Z}$, $F_n = A$ for all $n \leq 0$, and $F_{n+1} = F_1 F_n$ for all $n \gg 0$. In addition, if F_1 is an $\mathfrak{m}$ -primary ideal, then $\mathcal{F}$ is called a *Hilbert filtration* [4]. We write $G(\mathcal{F}) = \bigoplus_{n \geq 0} F_n / F_{n+1}$ for the associated graded ring of $\mathcal{F}$. In this chapter, we study the Gorenstein property of $G(\mathcal{F})$ with particular attention to the filtration $\{\bar{I}^n\}_{n \in \mathbb{Z}}$ where $\bar{J}$ denotes the integral closure of J . Let $\mathcal{F}$ be a Hilbert filtration. It is well known that the Hilbert function of $\mathcal{F}$, $H_{\mathcal{F}}(n) = \ell(A/F_n)$ where ℓ denotes the length, coincides with a polynomial of degree d for large n [53]. We denote this polynomial by $P_{\mathcal{F}}(n)$ and write it as

$$P_{\mathcal{F}}(n) = e_0(\mathcal{F}) \binom{n+d-1}{d} - e_1(\mathcal{F}) \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{F}). \quad (5.0.1)$$

Here $e_i(\mathcal{F})$ are integers called the *Hilbert coefficients* of $\mathcal{F}$. Recall that Q is said to be a *reduction* of $\mathcal{F}$ if $Q \subseteq F_1$ and $F_{n+1} = QF_n$ for $n \gg 0$. A reduction Q of $\mathcal{F}$ is said to be a *minimal reduction* of $\mathcal{F}$, if Q is a reduction of $\mathcal{F}$ and is minimal with respect to the inclusion. The reduction number of $\mathcal{F}$ with respect to a minimal reduction Q of $\mathcal{F}$,

$$r_Q(\mathcal{F}) = \min\{r \mid F_{n+1} = QF_n \text{ for all } n \geq r\},$$

and the *reduction number* of $\mathcal{F}$ is

$$r(\mathcal{F}) = \min\{r_Q(\mathcal{F}) \mid Q \text{ is a minimal reduction of } \mathcal{F}\}.$$

It is easy to observe that if I is an $\mathfrak{m}$ -primary ideal in A , then $\mathcal{F} = \{I^n\}_{n \in \mathbb{Z}}$ is a Hilbert filtration. In this case we write $G(I)$ for $G(\mathcal{F})$, $e_i(I)$ for $e_i(\mathcal{F})$, $r_Q(I)$ for $r_Q(\mathcal{F})$ and $r(I)$ for $r(\mathcal{F})$.

There is another interesting example of a filtration that appears in the study of resolution of singularities, namely $\mathcal{F} = \{\bar{I}^n\}_{n \in \mathbb{Z}}$.

In [60] Rees showed that if A is an analytically unramified local ring and I is an $\mathfrak{m}$ -primary ideal in A , then $\mathcal{F} = \{\bar{I}^n\}_{n \in \mathbb{Z}}$ is a Hilbert filtration. In this case we write $\bar{G}(I)$ for $G(\mathcal{F})$. We denote $\bar{e}_i(I)$ for $e_i(\mathcal{F})$, $\bar{r}_Q(I)$ for $r_Q(\mathcal{F})$, and $\bar{r}(I)$ for $r(\mathcal{F})$ and call them as the *normal Hilbert coefficients*, *normal reduction number with respect to a minimal reduction Q of $\{\bar{I}^n\}_{n \in \mathbb{Z}}$* and *normal reduction number*, respectively.

It has been observed that $G(I)$ and $\bar{G}(I)$ are rarely Cohen-Macaulay even if A is Cohen-Macaulay. A lot of attention in the literature is devoted to obtain criteria for $G(\mathcal{F})$ to be Cohen-Macaulay in terms of Hilbert coefficients and/or reduction number with particular attention to the filtration $\{I^n\}_{n \in \mathbb{Z}}$ and $\{\bar{I}^n\}_{n \in \mathbb{Z}}$. For instance, it is well-known that if $r(\mathcal{F}) \leq 1$ in a Cohen-Macaulay local ring, then $G(\mathcal{F})$ is Cohen-Macaulay [53]. Also, it is known that if $e_1(\mathcal{F}) = 0$ in a Cohen-Macaulay local ring, then $G(\mathcal{F})$ is Cohen-Macaulay [53]. For these reasons, a huge amount of research is devoted to obtain a sharp bound on the Hilbert coefficients and reduction number, and studying the behaviour of $G(\mathcal{F})$ when these bounds are extremal. This was also a theme studied by several mathematicians, for example, Huneke, Goto, Lipman, Sally among others.

In [29] Heinzer, Kim, and Ulrich established a criterion for Gorensteinness of $G(I)$ when A is an Artinian Gorenstein local ring. This has been extended for Hilbert filtrations in [30]. They proved that $G(\mathcal{F})$ is Gorenstein if and only if the h -vector of $G(\mathcal{F})$ is symmetric. In this chapter we extend this result for any d -dimensional Gorenstein local ring (Theorem 5.1.1). We use this result to characterize the Gorensteinness of $G(\mathcal{F})$ in terms of the Hilbert coefficients.

A first result in this direction is obtained in Section 5.1 when $r(\mathcal{F}) \leq 2$ (Theorem 5.1.6). We make an additional assumption when $r(\mathcal{F}) = 2$, namely, there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. This condition ensures that $G(\mathcal{F})$ is Cohen-Macaulay (Lemma 5.1.3), and helps to compute the Hilbert series of $G(\mathcal{F})$. Notice that this condition is not restrictive. For instance, the filtrations $\mathcal{F} = \{\bar{I}^n\}_{n \in \mathbb{Z}}$ and $\mathcal{F} = \{\mathfrak{m}^n\}_{n \in \mathbb{Z}}$ satisfy this criterion. As a consequence of our result, we obtain a criterion for Gorensteinness of $\bar{G}(I)$ (Corollary 5.1.7). This recovers and extends a recent result by Okuma, Watanabe, and Yoshida

in [58].

In Section 5.2, we next investigate the case $r(\mathcal{F}) > 2$. A difficulty in this case is the computation of the Hilbert series of $G(\mathcal{F})$. In order to compute this, we introduce the notion of *relative reduction number* $\text{nr}(\mathcal{F})$ and *maximal relative reduction number* for Hilbert filtration $\mathcal{F}$. This is inspired by the definition of relative reduction number for normal filtrations defined in [57] in dimension two.

The *relative reduction number* of $\mathcal{F}$ with respect to a minimal reduction Q of $\mathcal{F}$ is defined as

$$\text{nr}_Q(\mathcal{F}) = \min\{r \in \mathbb{Z}_+ \mid F_{r+1} = QF_r\}.$$

The *relative reduction number* of $\mathcal{F}$ is defined as

$$\text{nr}(\mathcal{F}) = \min\{\text{nr}_Q(\mathcal{F}) \mid Q \text{ is a minimal reduction of } \mathcal{F}\}.$$

Observe that $\text{nr}(\mathcal{F}) \leq r(\mathcal{F})$ and the inequality can be strict in general (see [55, Example 2.8]). If $\mathcal{F} = \{I^n\}_{n \in \mathbb{Z}}$, then $\text{nr}_Q(\mathcal{F}) = r_Q(\mathcal{F})$ for any minimal reduction Q of I and hence $\text{nr}(\mathcal{F}) = r(\mathcal{F})$. For $\mathcal{F} = \{\overline{I}^n\}_{n \in \mathbb{Z}}$, we denote $\text{nr}_Q(\mathcal{F})$ by $\text{nr}_Q(I)$ and $\text{nr}(\mathcal{F})$ by $\text{nr}(I)$ and the latter is called the *relative normal reduction number*. Note that $\text{nr}(I)$ does not refer to the relative reduction number of the filtration $\mathcal{F} = \{I^n\}_{n \in \mathbb{Z}}$.

We derive an upper bound (see Theorem 5.2.2) for $\text{nr}(\mathcal{F})$ in a d -dimensional Cohen-Macaulay local ring when $\text{depth } G(\mathcal{F}) \geq d - 1$. We prove that

$$\text{nr}(\mathcal{F}) \leq e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1.$$

In [55] the above inequality was proved for $\mathcal{F} = \{\overline{I}^n\}_{n \in \mathbb{Z}}$ in a two-dimensional normal excellent local domain. We also examine the conditions under which $\text{nr}(\mathcal{F})$ achieves the upper bound, which serves as a crucial technical assumption for the computation of the Hilbert series of $G(\mathcal{F})$. We say $\mathcal{F}$ has *maximal relative reduction number* if $\text{nr}(\mathcal{F}) = e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1$.

We conclude with some interesting examples of semigroup rings in dimension one in Section 5.3 and use our criteria to detect Gorensteinness of $\overline{G}(\mathfrak{m})$.

5.1 Gorensteinness of $G(\mathcal{F})$ in small reduction numbers

In this section, we discuss the Gorenstein property of $G(\mathcal{F})$ of a Hilbert filtration $\mathcal{F}$ in the case of $r(\mathcal{F}) \leq 2$.

In [30] Heinzer, Kim and Ulrich gave criteria for $G(\mathcal{F})$ to be Gorenstein for a Hilbert filtration $\mathcal{F}$ in an Artinian Gorenstein local ring. In the following theorem, we extend this result for any d -dimensional Gorenstein local ring. This plays a very important role in determining the Gorenstein property of the associated graded ring in terms of the Hilbert coefficients of $\mathcal{F}$.

Theorem 5.1.1. *Let $(A, \mathfrak{m})$ be a d -dimensional Gorenstein local ring, $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Suppose $G(\mathcal{F})$ is Cohen-Macaulay. Let*

$$\text{HS}_{G(\mathcal{F})}(z) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d} \quad (h_s \neq 0)$$

be the Hilbert series of $G(\mathcal{F})$. Then $G(\mathcal{F})$ is Gorenstein if and only if $h_{s-i} = h_i$ for all $i = 0, 1, \dots, s$.

Proof. Let Q be a minimal reduction of $\mathcal{F}$. By [35, Theorem 8.6.3] there exists a generating set $x_1, \dots, x_d$ of Q such that $x_1, \dots, x_d$ is a superficial sequence. Since $G(\mathcal{F})$ is Cohen-Macaulay, $x_1^*, \dots, x_d^*$ is a $G(\mathcal{F})$ -regular sequence by [33, Lemma 2.1] where $x_i^* = x_i + F_2$ for $i = 1, 2, \dots, d$. Therefore $G(\mathcal{F})/(x_1^*, \dots, x_d^*) \cong G(\mathcal{F}/Q)$. Since $x_1^*, \dots, x_d^*$ is a regular sequence in $G(\mathcal{F})$,

$$\text{HS}_{G(\mathcal{F})/(x_1^*, \dots, x_d^*)}(t) = h_0 + h_1 t + \cdots + h_s t^s.$$

Hence

$$\text{HS}_{G(\mathcal{F}/Q)}(t) = h_0 + h_1 t + \cdots + h_s t^s.$$

Therefore by [30, Theorem 4.2] $G(\mathcal{F}/Q)$ is Gorenstein if and only if $h_{s-i} = h_i$ for $i = 0, 1, \dots, s$. Now the result follows because $x_1^*, \dots, x_d^*$ is a regular sequence in $G(\mathcal{F})$ and $G(\mathcal{F})/(x_1^*, \dots, x_d^*) \cong G(\mathcal{F}/Q)$. $\square$

In general for an arbitrary filtration $\mathcal{F}$, we have $\text{nr}_Q(\mathcal{F}) \leq r_Q(\mathcal{F})$, and hence $\text{nr}(\mathcal{F}) \leq r(\mathcal{F})$. In [55, Example 2.8] the authors gave an example for which the inequality is strict for the filtration $\mathcal{F} = \{\overline{I^n}\}_{n \in \mathbb{Z}}$. In the following proposition we show that if $G(\mathcal{F})$ is Cohen-Macaulay, then $\text{nr}(\mathcal{F}) = r(\mathcal{F})$ for any Hilbert filtration $\mathcal{F}$.

Proposition 5.1.2. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d . Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ be a Hilbert filtration. If $G(\mathcal{F})$ is Cohen-Macaulay, then $\text{nr}_Q(\mathcal{F}) = r_Q(\mathcal{F})$ for any minimal reduction Q of $\mathcal{F}$. In particular, $\text{nr}(\mathcal{F}) = r(\mathcal{F})$.*

Proof. We always have $\text{nr}_Q(\mathcal{F}) \leq r_Q(\mathcal{F})$. By [35, Theorem 8.6.3] choose Q a minimal reduction of F_1 such that $\mu(Q) = d$. Since $G(\mathcal{F})$ is Cohen-Macaulay, by [33, Proposition

3.5] $Q \cap F_n = QF_{n-1}$ for all $n \geq 1$. Let $m = \text{nr}_Q(\mathcal{F})$. Then $F_{m+1} = QF_m$. Hence $F_{m+2} \subseteq F_{m+1} \subseteq Q$. Therefore

$$F_{m+2} = F_{m+2} \cap Q = QF_{m+1}.$$

Continuing this successively we get $F_{k+1} = QF_k$ for all $k \geq m$. This implies that $r_Q(\mathcal{F}) \leq m = \text{nr}_Q(\mathcal{F})$. Thus $r_Q(\mathcal{F}) = \text{nr}_Q(\mathcal{F})$. $\square$

Our goal is to use Theorem [5.1.1](#) to obtain criteria for $G(\mathcal{F})$ to be Gorenstein in terms of the Hilbert coefficients. For this, first of all we should know whether $G(\mathcal{F})$ is Cohen-Macaulay. There is a lot of research in the literature on obtain criteria for $G(\mathcal{F})$ to be Cohen-Macaulay in terms of numerical invariants such as Hilbert coefficients and reduction number. Below we present a well-known criteria for $G(\mathcal{F})$ to be Cohen-Macaulay in terms of reduction number. We remark that the condition in (b) is not restrictive. For instance, the filtrations $\mathcal{F} = \{\overline{I}^n\}_{n \in \mathbb{Z}}$ and $\mathcal{F} = \{\mathfrak{m}^n\}_{n \in \mathbb{Z}}$ satisfy this criterion.

Lemma 5.1.3. *Let $(A, \mathfrak{m})$ be a d -dimensional Cohen-Macaulay local ring and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Then $G(\mathcal{F})$ is Cohen-Macaulay in each of the following cases:*

- a) $r(\mathcal{F}) = 1$;
- b) *there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$.*

Proof. The proof follows by [\[33\]](#), Proposition 3.5]. $\square$

Now in order to provide conditions for $G(\mathcal{F})$ to be Gorenstein, we need to compute the Hilbert series of $G(\mathcal{F})$. The computation of the Hilbert series of $G(\mathcal{F})$ is not an easy task in general. In the following proposition we compute the Hilbert series of $G(\mathcal{F})$ in the case $\mathcal{F}$ satisfies the conditions (a), (b) of Lemma [5.1.3](#), and under certain additional assumptions when $r(\mathcal{F}) \geq 3$. For this, we recall the following definition: for a Hilbert filtration $\mathcal{F}$ the *postulation number* of $\mathcal{F}$ is given by

$$n(\mathcal{F}) = \sup\{n \in \mathbb{Z} \mid H_{\mathcal{F}}(n) \neq P_{\mathcal{F}}(n)\}.$$

If $\mathcal{F} = \{\overline{I}^n\}_{n \in \mathbb{Z}}$, then $n(\mathcal{F})$ is denoted by $\bar{n}(I)$. We also recall the result [\[53\]](#), Corollary 3.8] that will be used frequently in the remainder of the chapter. Let $(A, \mathfrak{m})$ be a d -dimensional Cohen-Macaulay local ring with infinite residue field and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration of ideals where F_1 is an $\mathfrak{m}$ -primary ideal. Suppose $\text{depth } G(\mathcal{F}) \geq d - 1$. Then

$$r(\mathcal{F}) = n(\mathcal{F}) + d. \tag{5.1.1}$$

We can use the following lemma in order to reduce several problems to the case of dimension one. Recall that if $f : \mathbb{Z} \rightarrow \mathbb{N}$ is a numerical function, then $\Delta(f) : \mathbb{Z} \rightarrow \mathbb{Z}$ is a numerical function defined by $\Delta(f)(n) = f(n) - f(n-1)$.

Lemma 5.1.4 ([33], Lemma 2.1). *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Let k be an integer with $1 \leq k \leq d$ and let $x_1, x_2, \dots, x_d \in F_1$ be a sequence such that*

(i) $Q = (x_1, x_2, \dots, x_d)$ is a minimal reduction of $\mathcal{F}$.

(ii) $\underline{x} = x_1, \dots, x_k$ is a superficial sequence in F_1 .

If $\text{depth } G(\mathcal{F}) \geq k$, then

(a) $x_1^*, \dots, x_k^*$ forms a $G(\mathcal{F})$ -regular sequence.

(b) If we put $\overline{\mathcal{F}} = \mathcal{F}/(\underline{x}) = \{F_n + (\underline{x})/(\underline{x})\}_{n \in \mathbb{Z}}$, then

$$P_{\overline{\mathcal{F}}}(n) = \Delta^k(P_{\mathcal{F}}(n)) = \sum_{i=0}^{d-k} e_i(\mathcal{F})(-1)^i \binom{n+d-k-1-i}{d-k-i}.$$

In particular, $e_i(\overline{\mathcal{F}}) = e_i(\mathcal{F})$ for each $i = 0, 1, \dots, d-k$.

Proposition 5.1.5. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Put $\lambda = \ell(A/F_1)$, $e_i = e_i(\mathcal{F})$ for $i \geq 0$.*

(a) If $r(\mathcal{F}) = 1$, then $e_1 = e_0 - \lambda$, $e_i = 0$ for all $i \geq 2$ and

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{\lambda + (e_0 - \lambda)t}{(1-t)^d} = \frac{(e_0 - e_1) + e_1 t}{(1-t)^d}.$$

(b) Suppose there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$.

Then $e_i = 0$ for $i \geq 3$ and

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{\lambda + (2e_0 - e_1 - 2\lambda)t + (\lambda - e_0 + e_1)t^2}{(1-t)^d}.$$

Also, if $d \geq 2$, then $e_2 = \lambda - e_0 + e_1$ and

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{(e_0 - e_1 + e_2) + (e_1 - 2e_2)t + e_2 t^2}{(1-t)^d}.$$

Proof. First, we prove (a) and (b) together for $d = 1$. (a) By Lemma 5.1.3 $G(\mathcal{F})$ is Cohen-Macaulay. Hence using Equation (5.1.1) we get $n(\mathcal{F}) = 1 - 1 = 0$. Hence for all $n \geq 1$, $H_{\mathcal{F}}(n) = P_{\mathcal{F}}(n) = e_0 n - e_1$. Taking $n = 1$, we obtain $\lambda = \ell(A/F_1) = e_0 - e_1$. Also, $\ell(F_n/F_{n+1}) = \ell(A/F_{n+1}) - \ell(A/F_n) = P_{\mathcal{F}}(n+1) - P_{\mathcal{F}}(n) = e_0$ for all $n \geq 1$. Therefore

$$\text{HS}_{G(\mathcal{F})}(t) = \sum_{n \in \mathbb{Z}} \ell(F_n/F_{n+1})t^n = \lambda + \sum_{n \geq 1} e_0 t^n = \frac{\lambda + (e_0 - \lambda)t}{(1-t)}.$$

This implies $e_i = 0$, for all $i \geq 2$.

(b) By Lemma 5.1.3 $G(\mathcal{F})$ is Cohen-Macaulay. Hence using Equation (5.1.1) $n(\mathcal{F}) = 2 - 1 = 1$. In particular, $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$ for all $n \geq 2$. Taking $n = 2$, we obtain $\ell(A/F_2) = 2e_0 - e_1$. Also, $\ell(F_n/F_{n+1}) = P_{\mathcal{F}}(n+1) - P_{\mathcal{F}}(n) = e_0$ for all $n \geq 2$. Thus the Hilbert series of $G(\mathcal{F})$ is given by

$$\begin{aligned} \text{HS}_{G(\mathcal{F})}(t) &= \sum_{n \in \mathbb{Z}} \ell(F_n/F_{n+1})t^n \\ &= \ell(A/F_1) + (\ell(A/F_2) - \ell(A/F_1))t + \sum_{n \geq 2} \ell(F_n/F_{n+1})t^n \\ &= \lambda + (2e_0 - e_1 - \lambda)t + \sum_{n \geq 2} e_0 t^n \\ &= \frac{\lambda + (2e_0 - e_1 - 2\lambda)t + (\lambda - e_0 + e_1)t^2}{1-t}. \end{aligned}$$

Next suppose that $d \geq 2$. By Lemma 5.1.3 $G(\mathcal{F})$ is Cohen-Macaulay. Then one can take a sequence $x_1, x_2, \dots, x_d$ which satisfies the condition of Lemma 5.1.4 for $k = d - 1$. If we put $\bar{A} = A/(\underline{x})$, then $\bar{A}$ is a one-dimensional Cohen-Macaulay local ring and $\bar{\mathcal{F}}$ is a Hilbert filtration. Also, $\ell(\bar{A}/\bar{F}_1) = \lambda$, $e_i(\bar{\mathcal{F}}) = e_i$ for $i = 0, 1$, and $r(\bar{\mathcal{F}}) = r(\mathcal{F})$. Indeed, the last assertion follows from the following fact:

$$\frac{\overline{F_{n+1}}}{x_d \bar{F}_n} = \frac{F_{n+1} + \underline{x}A}{QF_n + \underline{x}A} \cong \frac{F_{n+1}}{(QF_n + \underline{x}A) \cap F_{n+1}} = \frac{F_{n+1}}{QF_n + \underline{x}F_n} = \frac{F_{n+1}}{QF_n}. \quad (5.1.2)$$

Therefore one can reduce the proof to the case of dimension one.

Moreover, if $d \geq 2$, then since $n(\mathcal{F}) = 2 - d \leq 0$, we have $H_{\mathcal{F}}(1) = P_{\mathcal{F}}(1)$ and thus $\lambda = e_0 - e_1 + e_2$. Also, $e_i(\mathcal{F}) = 0$ for $i \geq 3$. $\square$

Using Proposition 5.1.5 we provide criteria for $G(\mathcal{F})$ to be Gorenstein in terms of the Hilbert coefficients.

Theorem 5.1.6. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 1$. Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ be a Hilbert filtration.*

(a) Let $r(\mathcal{F}) = 1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = 2e_1(\mathcal{F})$.

(b) Suppose there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = e_1(\mathcal{F})$.

Proof. Put $\lambda = \ell(A/F_1)$ and $e_i = e_i(\mathcal{F})$ for all $i \geq 0$. In both the cases, by Lemma 5.1.3 $G(\mathcal{F})$ is Cohen-Macaulay.

(a) By Proposition 5.1.5, we have

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{(e_0 - e_1) + e_1 t}{(1 - t)^d}.$$

Also, Proposition 5.1.5 we have $\lambda = \ell(A/F_1) = e_0 - e_1$. Moreover, $e_0 = \ell(A/Q)$. Hence $e_1 = \ell(F_1/Q) \neq 0$ as $r(\mathcal{F}) = 1$. Therefore $e_1 \neq 0$. Using Theorem 5.1.1, we obtain $G(\mathcal{F})$ is Gorenstein if and only if $e_0 = 2e_1$.

(b) By Proposition 5.1.5, we have

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{\lambda + (2e_0 - e_1 - 2\lambda)t + (\lambda - e_0 + e_1)t^2}{(1 - t)^d}.$$

By Theorem 5.1.1, we obtain $G(\mathcal{F})$ is Gorenstein if and only if $e_0 = e_1$. □

Now we recover the result in [58, Theorem 2.1]. Moreover, we extend it for any d -dimensional analytically unramified Gorenstein local ring.

Corollary 5.1.7. *Let $(A, \mathfrak{m})$ be a d -dimensional analytically unramified Gorenstein local ring of dimension $d \geq 1$, and I an $\mathfrak{m}$ -primary ideal in A .*

(a) If $\bar{r}(I) = 1$, then $\bar{G}(I)$ is Gorenstein if and only if $e_0(I) = 2\bar{e}_1(I)$.

(b) If $\bar{r}(I) = 2$, then $\bar{G}(I)$ is Gorenstein if and only if $e_0(I) = \bar{e}_1(I)$.

Proof. (a) If $\bar{r}(I) = 1$, then the result follows by Theorem 5.1.6.

(b) Let $\bar{r}(I) = 2$. By [42, Theorem 1], we obtain $Q \cap \bar{I}^2 = Q\bar{I}$. Hence by Theorem 5.1.6, the result follows. □

5.2 A Bound on relative reduction number

In this section we obtain an upper bound for the relative reduction number $\text{nr}(\mathcal{F})$ of $\mathcal{F}$ when the depth of $G(\mathcal{F})$ is almost maximal.

In general, the reduction number $r_Q(\mathcal{F})$ of $\mathcal{F}$ and the relative reduction number $\text{nr}_Q(\mathcal{F})$ of $\mathcal{F}$ are dependent on the minimal reduction Q . It is well-known that if $\text{depth } G(\mathcal{F}) \geq d - 1$, then $r_Q(\mathcal{F})$ is independent of the minimal reduction Q . In the following proposition we show that $\text{nr}_Q(\mathcal{F})$ is also independent of Q if $\text{depth } G(\mathcal{F}) \geq d - 1$.

Proposition 5.2.1. *Let $(A, \mathfrak{m})$ be a d -dimensional Cohen-Macaulay local ring, and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. If $\text{depth } G(\mathcal{F}) \geq d - 1$, then $\text{nr}_Q(\mathcal{F})$ is independent of any minimal reduction Q .*

Proof. Using [53, Theorem 3.6] for all $n \in \mathbb{Z}$, we have

$$\Delta^d(P_{\mathcal{F}}(n) - H_{\mathcal{F}}(n)) = \ell(F_{n+d}/QF_{n+d-1}).$$

Since the expression on the left-hand side is independent of Q , we get $\ell(F_{n+d}/QF_{n+d-1})$ is independent of Q . Hence it is easy to verify that $\text{nr}_Q(\mathcal{F})$ is independent of Q . $\square$

If $(A, \mathfrak{m})$ is an analytically unramified Cohen-Macaulay local ring of dimension two and $\mathcal{F} = \{\overline{I}^n\}_{n \in \mathbb{Z}}$, then $\text{depth } G(\mathcal{F}) \geq 1$ by [53, Proposition 3.25]. Hence by Proposition 5.2.1 $\text{nr}_Q(I)$ is independent of Q and thus $\text{nr}(I) = \text{nr}_Q(I)$. This is a reason that the authors use the notation $\text{nr}(I)$ (without suffix) for the relative normal reduction number in [55].

In the next theorem we extend a result by Okuma, Rossi, Watanabe, and Yoshida. In [55, Theorem 2.7] they showed that if $(A, \mathfrak{m})$ is an excellent two-dimensional normal local domain containing an algebraically closed field $A/\mathfrak{m}$, then for an $\mathfrak{m}$ -primary ideal I

$$\text{nr}(I) \leq \bar{e}_1(I) - \bar{e}_0(I) + \ell(A/\bar{I}) + 1.$$

They also provided a criterion for the equality to hold. In the following theorem we prove an analogue bound for any Hilbert filtration in a d -dimensional Cohen-Macaulay local ring, provided $G(\mathcal{F})$ has almost maximal depth.

Theorem 5.2.2. *Let $(A, \mathfrak{m})$ be a d -dimensional Cohen-Macaulay local ring, and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Suppose $\text{depth } G(\mathcal{F}) \geq d - 1$. Then*

- (a) $\text{nr}(\mathcal{F}) \leq e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1.$
- (b) *The equality holds in (1) if and only if $\text{nr}(\mathcal{F}) = r(\mathcal{F})$ and $\ell(F_n/QF_{n-1}) = 1$ for all $2 \leq n \leq \text{nr}(\mathcal{F})$ for any minimal reduction Q of $\mathcal{F}$.*

When this is the case, $e_i(\mathcal{F}) = \binom{\text{nr}(\mathcal{F})}{i}$ for $2 \leq i \leq d$.

A Hilbert filtration $\mathcal{F}$ is said to have a *maximal relative reduction number* (or, $\text{nr}(\mathcal{F})$ is *maximal*) if $\text{nr}(\mathcal{F}) = e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1$.

Proof. (a) Using [33, Proposition 4.6], for any minimal reduction Q of $\mathcal{F}$ we have,

$$\begin{aligned} e_1(\mathcal{F}) &= \sum_{n \geq 1}^{r(\mathcal{F})} \ell(F_n/QF_{n-1}) \\ &= \ell(F_1/Q) + \sum_{n \geq 2}^{r(\mathcal{F})} \ell(F_n/QF_{n-1}) \\ &= e_0(\mathcal{F}) - \ell(A/F_1) + \sum_{n \geq 2}^{r(\mathcal{F})} \ell(F_n/QF_{n-1}) \quad (\text{as } e_0(\mathcal{F}) = \ell(A/Q)). \end{aligned} \quad (5.2.1)$$

We observe that $\ell(F_n/QF_{n-1}) \geq 1$ for $2 \leq n \leq \text{nr}(\mathcal{F})$. Therefore by (5.2.1)

$$\text{nr}(\mathcal{F}) - 1 \leq \sum_{n \geq 2}^{r(\mathcal{F})} \ell(F_n/QF_{n-1}) = e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1), \quad (5.2.2)$$

and hence $\text{nr}(\mathcal{F}) \leq e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1$.

(b) Suppose $\text{nr}(\mathcal{F}) = e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1$. Then by (5.2.2) $\ell(F_n/QF_{n-1}) = 0$ for $n \geq \text{nr}_Q(\mathcal{F}) + 1$ and $\ell(F_n/QF_{n-1}) = 1$ for $2 \leq n \leq \text{nr}(\mathcal{F})$. Therefore $\text{nr}(\mathcal{F}) = r_Q(\mathcal{F})$, which yields $\text{nr}(\mathcal{F}) = r(\mathcal{F})$ as $r_Q(\mathcal{F})$ is independent of Q . The converse is also clear.

Again using [33, Proposition 4.6] we have, for $i \geq 2$,

$$e_i(\mathcal{F}) = \sum_{n \geq i}^{\text{nr}(\mathcal{F})} \ell(F_n/QF_{n-1}) \binom{n-1}{i-1} = \sum_{n \geq i}^{\text{nr}(\mathcal{F})} \binom{n-1}{i-1} = \binom{\text{nr}(\mathcal{F})}{i}.$$

□

Using Theorem 5.2.2 for the filtration $\mathcal{F} = \{\bar{I}^n\}_{n \in \mathbb{Z}}$ in dimension two, we recover and extend the result [55, Theorem 2.7] by Okuma, Rossi, Watanabe and Yoshida.

Corollary 5.2.3 ([55], Theorem 2.7). *Let $(A, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay analytically unramified local ring, and I an $\mathfrak{m}$ -primary ideal. Let Q a minimal reduction of I . Then*

$$\text{nr}(I) \leq \bar{e}_1(I) - e_0(I) + \ell(A/\bar{I}) + 1.$$

Moreover, the equality holds if and only if $\ell(\bar{I}^n/Q\bar{I}^{n-1}) = 1$ for all $2 \leq n \leq \text{nr}(I)$. In this case, $\text{nr}(I) = \bar{r}(I)$. Also, $\bar{e}_2(I) = \binom{\text{nr}(I)}{2}$.

Proof. Since $\text{depth } \overline{G}(I) \geq 1$ by [53, Proposition 3.25], the result follows from Theorem 5.2.2. $\square$

Remark 5.2.4. If $r(\mathcal{F}) = 1$, then $\text{nr}(\mathcal{F}) = r(\mathcal{F})$. Moreover, by Proposition 5.1.5 we have $\ell(A/F_1) = e_0 - e_1$. Hence $\text{nr}(\mathcal{F})$ is maximal.

Suppose $d \geq 2$ and there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. Then by Proposition 5.1.5, we obtain $\text{nr}(\mathcal{F})$ is maximal if and only if $e_2(\mathcal{F}) = 1$.

Now we compute the Hilbert series of $G(\mathcal{F})$ when $\text{nr}(\mathcal{F})$ is maximal.

Theorem 5.2.5. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$, and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Suppose $\text{depth } G(\mathcal{F}) \geq d - 1$, and $\text{nr}(\mathcal{F}) = r \geq 2$ is maximal. Put $\lambda = \ell(A/F_1)$ and $e_0 = e_0(\mathcal{F})$. Then*

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{\lambda + (e_0 - \lambda - 1)t + t^r}{(1 - t)^d}.$$

Proof. First suppose $d = 1$. Let $Q = (x)$ a minimal reduction of $\mathcal{F}$. By Theorem 5.2.2 we have $r(\mathcal{F}) = \text{nr}(\mathcal{F})$ and $\ell(F_n/QF_{n-1}) = 1$ for all $2 \leq n \leq r(\mathcal{F})$. Observe that for $1 \leq n \leq r - 1$,

$$\begin{aligned} \ell(F_n/F_{n+1}) &= \ell(A/F_{n+1}) - \ell(A/F_n) \\ &= \ell(A/xF_n) - \ell(F_{n+1}/xF_n) - \ell(A/F_n) \\ &= \ell(A/xA) + \ell(A/F_n) - \ell(F_{n+1}/xF_n) - \ell(A/F_n) \\ &= e_0 - 1. \end{aligned}$$

For $n \geq r$,

$$\begin{aligned} \ell(F_n/F_{n+1}) &= \ell(A/F_{n+1}) - \ell(A/F_n) \\ &= (n + 1)e_0 - e_1 - (ne_0 - e_1) = e_0. \end{aligned}$$

Therefore

$$\begin{aligned} \text{HS}_{G(\mathcal{F})}(t) &= \sum_{n \geq 0} \ell(F_n/F_{n+1})t^n \\ &= \lambda + \sum_{n=1}^{r-1} (e_0 - 1)t^n + \sum_{n \geq r} e_0 t^n \\ &= \lambda + (e_0 - 1) \frac{t(1 - t^{r-1})}{1 - t} + \frac{e_0 t^r}{1 - t} \end{aligned}$$

$$= \frac{\lambda + (e_0 - \lambda - 1)t + t^r}{1 - t}.$$

Next suppose $d \geq 2$. We take a sequence $x_1, x_2, \dots, x_d$ in F_1 which satisfies the condition of Lemma 5.1.4. Put $Q = (x_1, x_2, \dots, x_d)A$, $\underline{x} = x_1, \dots, x_{d-1}$, $\bar{A} = A/(\underline{x})A$, and $\bar{\mathcal{F}} = \mathcal{F}/(\underline{x})$. Then by Proposition 5.1.2 $\text{nr}(\bar{\mathcal{F}})$ is also maximal. Hence it follows from the argument as above that

$$\text{HS}_{G(\bar{\mathcal{F}})}(t) = \frac{\lambda + (e_0 - \lambda - 1)t + t^r}{1 - t}$$

and thus

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{\text{HS}_{G(\bar{\mathcal{F}})}(t)}{(1 - t)^{d-1}} = \frac{\lambda + (e_0 - \lambda - 1)t + t^r}{(1 - t)^d}.$$

□

As a consequence, we provide criteria for $G(\mathcal{F})$ to be Gorenstein in terms of the Hilbert coefficients in the case where $\text{nr}(\mathcal{F})$ is maximal.

Theorem 5.2.6. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 1$, and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Suppose that $\text{nr}(\mathcal{F}) = r \geq 2$ is maximal and $G(\mathcal{F})$ is Cohen-Macaulay. Then*

- (a) *When $r = 2$, $G(\mathcal{F})$ is Gorenstein if and only if $F_1 = \mathfrak{m}$.*
- (b) *When $r \geq 3$, $G(\mathcal{F})$ is Gorenstein if and only if $F_1 = \mathfrak{m}$ and $e_0(\mathfrak{m}) = 2$.*

Proof. (a) By Theorem 5.2.5, we have

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{\lambda + (e_0 - \lambda - 1)t + t^2}{(1 - t)^d}.$$

Thus, by Theorem 5.1.1, $G(\mathcal{F})$ is Gorenstein if and only if $\lambda = 1$, that is, $F_1 = \mathfrak{m}$.

(b) By Theorem 5.2.5, we have

$$\text{HS}_{G(\mathcal{F})}(t) = \frac{\lambda + (e_0 - \lambda - 1)t + t^r}{(1 - t)^d}.$$

Thus, by Theorem 5.1.1, $G(\mathcal{F})$ is Gorenstein if and only if $\lambda = 1$ and $e_0 - \lambda - 1 = 0$, that is, $F_1 = \mathfrak{m}$ and $e_0(\mathfrak{m}) = 2$. □

Applying Theorem 5.2.6 for the filtration $\{\bar{I}^n\}_{n \in \mathbb{Z}}$ we get the following result.

Corollary 5.2.7. *Let $(A, \mathfrak{m})$ be a analytically unramified Gorenstein local ring of dimension $d \geq 1$, and I an $\mathfrak{m}$ -primary ideal in A . Suppose that $r = \bar{r}(I) = \text{nr}(I)$ is maximal and $\bar{G}(I)$ is Cohen-Macaulay.*

- (a) When $\bar{r}(I) = 2$, $\bar{G}(I)$ is Gorenstein if and only if $\bar{I} = \mathfrak{m}$.
- (b) When $\bar{r}(I) \geq 3$, $\bar{G}(I)$ is Gorenstein if and only if $\bar{I} = \mathfrak{m}$ and $e_0(\mathfrak{m}) = 2$.

Proof. Follows from Theorem 5.2.6. □

5.3 Examples in semigroups

Note that if $(A, \mathfrak{m})$ is a positive dimensional Cohen-Macaulay local ring and I is an $\mathfrak{m}$ -primary ideal, then $\text{depth } \bar{G}(I) \geq 1$ by [53, Proposition 3.25]. In particular, if $\dim A = 1$, then $\bar{G}(I)$ is Cohen-Macaulay.

Next, we apply our results to one-dimensional semigroup rings.

Definition 5.3.1. Let $m_1 < m_2 < \dots < m_n$ be positive integers with $\gcd(m_1, m_2, \dots, m_n) = 1$, and S be the semigroup generated by $m_1, m_2, \dots, m_n$. Then the Fröbenius number $F(S)$ of S is defined as the smallest integer k such that $n \in S$ for all $n > k$.

A semigroup S is *symmetric* if $F(S)$ is odd and for every $a \in \mathbb{Z}$, either $a \in S$ or $F(S) - a \in S$.

Lemma 5.3.2. Let $A = K[[t^{m_1}, t^{m_2}, \dots, t^{m_n}]]$ be a semigroup ring of the semigroup generated by $m_1 < m_2 < \dots < m_n$ with $\gcd(m_1, m_2, \dots, m_n) = 1$. Then

$$\bar{r}(\mathfrak{m}) = \left\lceil \frac{F(S)}{m_1} \right\rceil.$$

Proof. We note that $Q = (t^{m_1})$ is a minimal reduction of $\mathfrak{m} = (t^{m_1}, t^{m_2}, \dots, t^{m_n})$, the maximal ideal of A . By [30, Example 7.3] we have $\bar{\mathfrak{m}}^i = \{\alpha \in A \mid \text{ord}(\alpha) \geq m_1 i\}$ where $\text{ord}(\alpha)$ is the degree of the initial form of α . Indeed, the natural inclusion map $A \rightarrow K[[t]]$ is an integral extension. We have $\bar{I} = \overline{IK[[t]]} \cap A$ for any ideal I in A . Since $K[[t]]$ is a PID, $(t^{m_1})K[[t]]$ is integrally closed. As $Q = (t^{m_1})$ is a minimal reduction of $\mathfrak{m}$, $\bar{Q}^i = \bar{\mathfrak{m}}^i$ for all i . Let k be the smallest integer such that $m_1 k > F(S)$.

Consider $\bar{\mathfrak{m}}^k = (t^{m_1 k}, t^{m_1 k+1}, \dots)$. Hence

$$Q\bar{\mathfrak{m}}^k = (t^{m_1(k+1)}, t^{m_1(k+1)+1}, \dots) = \bar{\mathfrak{m}}^{k+1}.$$

Therefore $\bar{r}(\mathfrak{m}) \leq k$.

By the definition of k we have $m_1(k-1) \leq F(S)$. Hence $F(S) + m_1 \geq m_1 k$. Therefore $t^{F(S)+m_1} \in \bar{\mathfrak{m}}^k$. Suppose $t^{F(S)+m_1} \in Q\bar{\mathfrak{m}}^{k-1}$. Then $t^{F(S)+m_1} = t^{m_1} b$ where $b \in A$. Since

t^{m_1} is a regular element in $k[[t]]$, we obtain $t^{F(S)} = b \in A$. This is a contradiction. Hence $t^{F(S)+m_1} \in \overline{m^k} \setminus Q\overline{m^{k-1}}$. Therefore $\bar{r}(\mathfrak{m}) = k$. $\square$

Example 5.3.3. Let $A = K[[t^3, t^4]]$ be a semigroup ring generated by t^3 and t^4 . The Fröbenius number $F(S)$ of the semigroup $S = \mathbb{N}3 + \mathbb{N}4$ is 5. Since S is symmetric, A is Gorenstein with $e_0(\mathfrak{m}) = 3$. Also, $\overline{m^i} = \{\alpha \in A \mid \text{ord}(\alpha) \geq 3i\}$. Note that $Q = (t^3)$ is a minimal reduction of $\mathfrak{m}$. Hence by Lemma 5.3.2 $\bar{r}(\mathfrak{m}) = 2$. Using Equation (5.1.1), $\bar{n}(\mathfrak{m}) = 2 - 1 = 1$. Hence $\bar{P}_{\mathfrak{m}}(n) = \bar{H}_{\mathfrak{m}}(n)$ for all $n \geq 2$. Therefore $\bar{e}_1(m) = 2e_0(\mathfrak{m}) - \ell(A/\overline{m^2}) = 2 \cdot 3 - 3 = 3$. Since $e_0(\mathfrak{m}) = \bar{e}_1(\mathfrak{m})$, by Corollary 5.1.7, $\bar{G}(\mathfrak{m})$ is Gorenstein.

Example 5.3.4. Let $A = K[[t^2, t^m]]$ be a semigroup ring generated by t^2 and t^m , where m is an odd integer ≥ 3 . The Fröbenius number $F(S)$ of the semigroup $S = \mathbb{N}2 + \mathbb{N}m$ is $m - 2$. Since S is symmetric, A is Gorenstein. Also, $\overline{m^i} = \{\alpha \in A \mid \text{ord}(\alpha) \geq 2i\}$. Note that $Q = (t^2)$ is a minimal reduction of $\mathfrak{m}$. Hence by Lemma 5.3.2 $\bar{r}(\mathfrak{m}) = (m-1)/2$. Therefore $\bar{P}_{\mathfrak{m}}(n) = \bar{H}_{\mathfrak{m}}(n)$ for all $n \geq (m-1)/2$. Also, since $e_0(\mathfrak{m}) = 2$, we have

$$\bar{e}_1(\mathfrak{m}) = e_0(\mathfrak{m})((m-1)/2) - \ell(A/\overline{m^{(m-1)/2}}) = (m-1) - (m-1)/2 = (m-1)/2.$$

Hence $\bar{r}(\mathfrak{m}) = \bar{e}_1(\mathfrak{m}) - e_0(\mathfrak{m}) + \ell(A/\overline{m}) + 1$ and thus $\text{nr}(\mathfrak{m})$ is maximal. Therefore, by Corollary 5.2.7, $\bar{G}(\mathfrak{m})$ is Gorenstein.

Example 5.3.5. Let $A = K[[t^4, t^5, t^6]]$ be a semigroup ring generated by 4, 5 and 6. The Fröbenius number of the semigroup $F(S) = 7$. Since S is symmetric, A is Gorenstein. By Lemma 5.3.2 we get $\bar{r}(\mathfrak{m}) = 2$. Using similar arguments as above we obtain $e_0(\mathfrak{m}) = 4$ and $\bar{e}_1(\mathfrak{m}) = 4$. Hence, by Corollary 5.1.7, $\bar{G}(\mathfrak{m})$ is Gorenstein.

The following example is from [30, Example 7.3]. They prove that $\bar{G}(\mathfrak{m})$ is not Gorenstein. We also prove the same by using our techniques.

Example 5.3.6. Let $A = K[[t^4, t^6, t^7]]$ be a semigroup ring generated by 4, 6 and 7. The Fröbenius number of the semigroup $F(S) = 9$ and the semigroup is symmetric. Hence A is a one-dimensional Gorenstein local ring. Let $\mathfrak{m} = (t^4, t^6, t^7)$. Then $\overline{m^i} = \{\alpha \in A \mid \text{ord}(\alpha) \geq 4i\}$, and $Q = (t^4)$ is a minimal reduction of $\mathfrak{m}$ with reduction number $\bar{r}(\mathfrak{m}) = 3$ by Lemma 5.3.2. By Equation (5.1.1), $\bar{n}(\mathfrak{m}) = 3 - 1 = 2$. Hence $\bar{P}_{\mathfrak{m}}(n) = \bar{H}_{\mathfrak{m}}(n)$ for all $n \geq 3$. In particular, $\bar{e}_1(\mathfrak{m}) = 3e_0(\mathfrak{m}) - \ell(A/\overline{m^3}) = 3 \cdot 4 - 7 = 5$. Then since $\text{nr}(\mathfrak{m}) = 3 = \bar{e}_1(\mathfrak{m}) - e_0(\mathfrak{m}) + \ell(A/\mathfrak{m}) + 1$ is maximal, but $e_0(\mathfrak{m}) = 4$, by Theorem 5.2.6 $\bar{G}(\mathfrak{m})$ is not Gorenstein.

5.4 Gorensteinness of $\overline{G}(\mathfrak{m})$ in Zariski hypersurface rings

We consider the following setup for the rest of the section. Let K be an algebraically closed field, and let $m \geq 1$, $2 \leq a \leq b$ be integers, and put $S = K[[x, y_1, \dots, y_m]]$ be a formal power series ring with $(m + 1)$ -variables over K . Then S is a complete regular local ring with the unique maximal ideal $\mathfrak{m}_S = (x, y_1, \dots, y_m)$. Put $\underline{y} = y_1, \dots, y_m$. Let g, f be polynomials such that

$$\begin{aligned} g &= g(\underline{y}) \in (\underline{y})^b \setminus (\underline{y})^{b+1}, \\ f &= f(x, \underline{y}) = x^a - g(\underline{y}). \end{aligned}$$

Then a hypersurface $A = S/(f)$ is called “Zariski-type hypersurface” if $a \neq 0$ in K . Let $Q = (\underline{y})A$. Note that Q is a minimal reduction of $\mathfrak{m} = \mathfrak{m}_S A$. Indeed, $\mathfrak{m}^a = Q\mathfrak{m}^{a-1}$. In this section, we compute $\bar{r}(\mathfrak{m})$ and prove the Cohen-Macaulayness of $\overline{G}(\mathfrak{m})$. In order to do that, we generalize several results for Brieskorn hypersurfaces in [57, Section 3] to that of Zariski-type hypersurfaces. We fix the following notation for the rest of the paper:

$$d = \gcd(a, b), \quad a' = \frac{a}{d}, \quad b' = \frac{b}{d} \quad \text{and} \quad n_k = \lfloor \frac{kb}{a} \rfloor \quad \text{for } k = 1, 2, \dots, a-1.$$

For all $n \geq 1$, we define the following two ideals:

$$\begin{aligned} J_n &:= Q^n + xQ^{n-n_1} + x^2Q^{n-n_2} + \dots + x^{a-1}Q^{n-n_{a-1}}, \\ I_n &:= (x^k y_1^{i_1} \dots y_m^{i_m} \mid kb' + (i_1 + \dots + i_m)a' \geq n, k, i_1, \dots, i_m \in \mathbb{Z}_{\geq 0}) A. \end{aligned}$$

Proposition 5.4.1. $J_n = I_{na'}$ for every $n \geq 1$.

Proof. First we prove $J_n \subset I_{na'}$. It is enough to show $x^k Q^{n-n_k} \subset I_{na'}$ for every $k = 1, 2, \dots, a-1$. For each k with $1 \leq k \leq a-1$, if $i_1 + \dots + i_m \geq n - n_k$, then

$$kb' + (i_1 + \dots + i_m)a' \geq kb' + (n - n_k)a' \geq kb' + \left(n - \frac{kb'}{a'}\right)a' = na',$$

as required. Next we prove the converse. If $x^k y_1^{i_1} \dots y_m^{i_m} \in I_{na'}$, then $kb' + (i_1 + \dots + i_m)a' \geq na'$. Thus

$$n_k = \lfloor \frac{kb'}{a'} \rfloor \geq n - (i_1 + \dots + i_m), \quad \text{that is, } i_1 + \dots + i_m \geq n - n_k.$$

Hence $x^k y_1^{i_1} \dots y_m^{i_m} \in x^k Q^{n-n_k}$. This yields $I_{na'} \subset J_n$. □

Proposition 5.4.2. Under the notation above,

(a) For every $k, n \geq 1$, $x^k \in \overline{Q^n}$ if and only if $n \leq n_k$.

(b) $Q^n \subset J_n \subset \overline{Q^n}$, that is, $\overline{J_n} = \overline{Q^n}$ for each $n \geq 1$.

Proof. (a) We first show that if $n \leq n_k$ then $x^k \in \overline{Q^n}$. By definition of n_k , we get

$$(x^k)^a = x^{ka} \in (Q^b)^k = Q^{kb} \subset Q^{an_k} = (Q^{n_k})^a.$$

Hence $x^k \in \overline{Q^{n_k}} \subset \overline{Q^n}$.

Suppose $x^k \in \overline{Q^n}$. We will show $n \leq n_k$. By assumption, there exists a nonzero divisor $c \in A$ such that $c(x^k)^\ell \in Q^{n\ell}$ for sufficiently large ℓ . By Artin-Rees' lemma, we can take an integer $\ell_0 \geq 1$ such that if $\ell \geq 0$, then $Q^\ell \cap cA = c(Q^\ell : c) \subset cQ^{\ell-\ell_0}$. Thus $Q^\ell : c \subset Q^{\ell-\ell_0}$ because c is a nonzero divisor.

Now suppose that $n \geq n_k + 1$. By the choice of n_k , we have

$$\frac{kb}{a} + \frac{1}{a} \leq n_k + 1 \leq n.$$

Then $kb + 1 \leq a(n_k + 1) \leq an$. If $\ell \geq \ell_0 + 1$, then

$$g^{k\ell} = x^{ak\ell} \in Q^{an\ell} : c \subset Q^{an\ell-\ell_0} \subset Q^{(kb+1)\ell-\ell_0} \subset Q^{kb\ell+1}.$$

Moreover, since $g \in Q^b \setminus Q^{b+1}$, if necessary, by replacing $\underline{y} = y_1, y_2, \dots, y_m$ with the other minimal generators of Q , we may assume that $g \equiv y_1^b \pmod{(y_2, \dots, y_m)A}$. Hence $y_1^{bk\ell} \in (y_1^{kb\ell+1}, y_2, \dots, y_m)$. This contradicts the fact that $y_1, y_2, \dots, y_m$ forms a regular sequence on A .

(b) It is enough to show $x^k y_1^{i_1} \dots y_m^{i_m} \in \overline{Q^n}$ if and only if $i_1 + \dots + i_m \geq n - n_k$. Since $\overline{G}(Q) \cong \overline{\mathcal{R}'}(Q)/t^{-1}\overline{\mathcal{R}'}(Q)$ and $\overline{\mathcal{R}'}(Q)$ is normal, we have $\text{depth} \overline{G}(Q) \geq 1$. Hence we obtain $\overline{Q^n} : y_j = \overline{Q^{n-1}}$ for every n, j . So one can easily see that

$$x^k y_1^{i_1} \dots y_m^{i_m} \in \overline{Q^n} \iff x^k \in \overline{Q^{n-(i_1+\dots+i_m)}}.$$

By (a), the last condition means that $i_1 + \dots + i_m \geq n - n_k$. Hence we get the required inclusions. $\square$

Set $\mathcal{I} = \{I_n\}_{n \in \mathbb{Z}}$, which is a filtration of ideals in A introduced above and, $I_n = A$ for every integer $n \leq 0$. We put

$$\mathcal{R}'(\mathcal{I}) = \bigoplus_{n \in \mathbb{Z}} I_n t^n \subset A[t, t^{-1}], \quad G(\mathcal{I}) = \bigoplus_{n \geq 0} I_n / I_{n+1} \cong \mathcal{R}'(\mathcal{I}) / t^{-1} \mathcal{R}'(\mathcal{I}).$$

In the following, we set $\underline{y} t^k = y_1 t^k, \dots, y_m t^k$ and $\underline{Y} = Y_1, \dots, Y_m$.

Lemma 5.4.3. Write $g = g_b + g_{b+1} + \cdots + g_c$, where g_i denotes a homogeneous polynomial of degree $i = b, b+1, \dots, c$. Then $\mathcal{R}'(\mathcal{I}) \cong A[xt^{b'}, \underline{y}t^{a'}, t^{-1}]$ and $G(\mathcal{I}) \cong K[X, \underline{Y}]/(X^a - g_b(\underline{Y}))$.

Proof. We have

$$\begin{aligned} \mathcal{R}'(\mathcal{I}) &\cong \bigoplus_{n \in \mathbb{Z}} I_n t^n \\ &\cong \sum_{k, i_1, \dots, i_m \in \mathbb{Z}_{\geq 0}} K x^k y_1^{i_1} \cdots y_m^{i_m} t^{kb' + (i_1 + \cdots + i_m)a'} + \sum_{n \geq 0} A t^{-n} \\ &= \sum_{k, i_1, \dots, i_m \in \mathbb{Z}_{\geq 0}} K (xt^{b'})^k (y_1 t^{a'})^{i_1} \cdots (y_m t^{a'})^{i_m} + \sum_{n \geq 0} A t^{-n} \\ &= A[xt^{b'}, y_1 t^{a'}, \dots, y_m t^{a'}, t^{-1}]. \end{aligned}$$

Put $X = xt^{b'}$, $Y_i = y_i t^{a'}$ for $i = 1, 2, \dots, m$ and $U = t^{-1}$. Then

$$\begin{aligned} X^a &= x^a t^{ab'} = x^a t^{a'b} \\ &= g_b(y_1, \dots, y_m) t^{a'b} + g_{b+1}(y_1, \dots, y_m) t^{a'b} + \cdots + g_c(y_1, \dots, y_m) t^{a'b} \\ &= g_b(Y_1, Y_2, \dots, Y_m) + g_{b+1}(Y_1, Y_2, \dots, Y_m) t^{-a'} + \cdots + g_c(Y_1, Y_2, \dots, Y_m) t^{-(c-b)a'} \\ &= g_b(Y_1, Y_2, \dots, Y_m) + g_{b+1}(Y_1, Y_2, \dots, Y_m) U^{a'} + \cdots + g_c(Y_1, Y_2, \dots, Y_m) U^{(c-b)a'}. \end{aligned}$$

Hence

$$\begin{aligned} G(\mathcal{I}) &\cong \mathcal{R}'(\mathcal{I})/t^{-1}\mathcal{R}'(\mathcal{I}) \\ &\cong k[X, Y_1, Y_2, \dots, Y_m]/(X^a - g_b(\underline{Y})). \end{aligned}$$

□

The normality of A is needed in the following proposition.

Proposition 5.4.4. Suppose that A is a Zariski-type hypersurface. Under the notation as above,

- (a) $G(\mathcal{I})$ is reduced.
- (b) $\overline{I_n} = I_n$ for every $n \geq 1$.
- (c) $\overline{J_n} = J_n$ for every $n \geq 1$.

Proof. (a) By Lemma 5.4.3, since $G(\mathcal{I}) \cong K[X, \underline{Y}]/(X^a - g_b(\underline{Y}))$ is a hypersurface, it satisfies the Serre condition (S_1) .

Put $f = X^a - g_b(\underline{Y})$. Then

$$\text{ht} \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y_1}, \dots, \frac{\partial f}{\partial Y_m}, f \right) \geq \text{ht}(X, g_b(\underline{Y})) \geq 1.$$

Hence $G(\mathcal{I})$ satisfies (R_0) and it is reduced.

(b) It follows from [52, Lemma 3.1].

(c) Since $J_n = I_{n\alpha'}$ for every $n \geq 1$ by Proposition 5.4.1, J_n is also integrally closed by (b). $\square$

The following theorem generalizes [57, Theorem 3.1].

Theorem 5.4.5. *Under the notation as above,*

$$(a) \quad \overline{\mathfrak{m}}^n = \overline{Q}^n = Q^n + xQ^{n-n_1} + x^2Q^{n-n_2} + \cdots + x^{a-1}Q^{n-n_{a-1}}.$$

$$(b) \quad \bar{r}_Q(\mathfrak{m}) = \text{nr}_Q(\mathfrak{m}) = n_{a-1} = \lfloor \frac{(a-1)b}{a} \rfloor.$$

(c) $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay.

Proof. (a) It follows from Proposition 5.4.2(b) and Proposition 5.4.4(c).

(b) It immediately follows from (a).

(c) By Proposition 5.4.4, $G(\mathcal{I})$ is a hypersurface. Therefore it is Cohen-Macaulay. Hence $\mathcal{R}'(\mathcal{I})$ is Cohen-Macaulay.

On the other hand, $\overline{\mathcal{R}'}(\mathfrak{m}) = \mathcal{R}'(\{J_n\}) = \mathcal{R}'(\{I_{na'}\})$ by Propositions 5.4.1, 5.4.2(b) and 5.4.4(c). Then $\overline{\mathcal{R}'}(\mathfrak{m})$ is also Cohen-Macaulay since $\overline{\mathcal{R}'}(\mathfrak{m})$ is isomorphic to the a' th Veronese subring of $\mathcal{R}'(\mathcal{I})$. Therefore $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay. $\square$

Next, we give a criterion for $\overline{G}(\mathfrak{m})$ to be Gorenstein in the case A is a Zariski-type hypersurface. First, we need the following criterion from [56]. We put $L_n = Q + \overline{\mathfrak{m}}^n$ and $\ell_n = \ell(A/L_n)$ for every $n \geq 1$.

Lemma 5.4.6 ([56], Proposition 2.5). *Put $r = r_Q(\mathfrak{m})$. Then the following conditions are equivalent:*

(a) $\overline{G}(\mathfrak{m})$ is Gorenstein.

(b) $Q: L_n = L_{r+1-n}$ for every $n = 1, 2, \dots, r$.

(c) $\ell_n + \ell_{r+1-n} = a$ for every $n = 1, 2, \dots, \lfloor \frac{r}{2} \rfloor$.

As an application of Proposition 5.4.4, we obtain the following proposition.

Proposition 5.4.7. *Under the same notation as above, we have*

$$\ell_n = \min\{k \in \mathbb{Z}_+ \mid n \leq n_k\}.$$

Proof. Suppose $n > n_{k-1}$ and $n \leq n_k$. Then

$$\begin{aligned} L_n &= Q + \overline{\mathfrak{m}^n} \\ &= Q + (Q^n + xQ^{n-n_1} + \cdots + x^{k-1}Q^{n-n_{k-1}} + x^kQ^{n-n_k} + \cdots) \\ &= Q + (x^k). \end{aligned}$$

Hence $\ell_n = \ell(K[x]/(x^k)) = k$. □

The following theorem is the main result in this section. Recall $d = \gcd(a, b)$.

Theorem 5.4.8. *$\overline{G}(\mathfrak{m})$ is Gorenstein if and only if $b \equiv 0, d \pmod{a}$. Moreover,*

- (a) *If $b \equiv 0 \pmod{a}$, then $\overline{G}(\mathfrak{m})$ is a reduced, hypersurface.*
- (b) *If $b \equiv d \pmod{a}$ and $d = 1$, then $\overline{G}(\mathfrak{m})$ is a non-reduced, hypersurface.*
- (c) *If $b \equiv d \pmod{a}$ and $d > 1$, then $\overline{G}(\mathfrak{m})$ is a complete intersection but not a hypersurface.*

In what follows, we prove the above theorem in several steps. By Theorem 5.4.5, we have

$$\begin{aligned} \mathfrak{m}^n t^n &= (Q^n + xQ^{n-n_1} + \cdots + x^{a-1}Q^{n-n_{a-1}})t^n \\ &= (Qt)^n + (xt^{n_1})(Qt)^{n-n_1} + \cdots + (x^{a-1}t^{n_{a-1}})(Qt)^{n-n_{a-1}}. \end{aligned}$$

Thus

$$\overline{\mathcal{R}'}(\mathfrak{m}) = K[\underline{xt}, \underline{yt}, \{x^k t^{n_k}\}_{1 \leq k \leq a-1}, t^{-1}].$$

Using this fact, we first consider the case where a divides b .

Proposition 5.4.9. *If a divides b , then $\overline{G}(\mathfrak{m})$ is a reduced hypersurface, and thus Gorenstein. In fact,*

$$\overline{G}(\mathfrak{m}) \cong K[X, \underline{Y}]/(X^a - g_b(\underline{Y})).$$

Proof. As $b = b'a$ for some integer $b' \geq 1$, we have $n_k = \lfloor kb' \rfloor = kb'$ for every $k = 1, 2, \dots, a-1$.

1. Hence one can easily see that

$$\overline{\mathcal{R}'}(\mathfrak{m}) = K[\underline{xt}, \underline{yt}, \{x^k t^{n_k}\}_{1 \leq k \leq a-1}, t^{-1}]$$

$$\begin{aligned}
&= K[xt^{n_1}, \underline{y}t, t^{-1}] \\
&\cong K[X, \underline{Y}, U]/(X^a - g_b(\underline{Y}) - \cdots - g_c(\underline{Y})U^{c-b}),
\end{aligned}$$

where $X = xt^{n_1}$, $Y_i = y_it$ ($i = 1, 2, \dots, m$), and $U = t^{-1}$.

Moreover, the claims on $G(\mathfrak{m})$ immediately follow from the fact that

$$\overline{G}(\mathfrak{m}) \cong \overline{\mathcal{R}'}(\mathfrak{m})/t^{-1}\overline{\mathcal{R}'}(\mathfrak{m}).$$

□

Secondly, we consider the case of $d = 1$ and $b \equiv 1 \pmod{a}$.

Proposition 5.4.10. *If $d = 1$ and $b = n_1a + 1$, then $\overline{G}(\mathfrak{m})$ is a non-reduced hypersurface, and thus Gorenstein. In fact,*

$$\overline{G}(\mathfrak{m}) \cong K[X, \underline{Y}]/(X^a) \cong G(\mathfrak{m}).$$

Proof. For every k with $1 \leq k \leq a - 1$, we have

$$n_k = \lfloor \frac{kb}{a} \rfloor = \lfloor \frac{k(n_1a + 1)}{a} \rfloor = kn_1 + \lfloor \frac{k}{a} \rfloor = kn_1.$$

Hence

$$\begin{aligned}
\overline{\mathcal{R}'}(\mathfrak{m}) &= K[xt, \underline{y}t, \{x^k t^{n_k}\}_{1 \leq k \leq a-1}, t^{-1}] \\
&= K[xt, \underline{y}t, xt^{n_1}, t^{-1}] \\
&\cong K[X, \underline{Y}, U]/(X^a - g_b(\underline{Y})U - \cdots - g_c(\underline{Y})U^{c-b+1}).
\end{aligned}$$

Indeed, put $X = xt^{n_1}$, $Y_i = y_it$ for $i = 1, 2, \dots, m$ and $U = t^{-1}$. Then $xt = (xt^{n_1})(t^{-1})^{n_1-1}$ and

$$\begin{aligned}
X^a = (xt^{n_1})^a &= (g_b + g_{b+1} + \cdots + g_c)t^{an_1} \\
&= (g_b + g_{b+1} + \cdots + g_c)t^{b-1} \\
&= (g_b t^b)U + g_{b+1}t^{b+1}U^2 + \cdots + g_c t^c U^{c-b+1} \\
&= g_b(\underline{Y})U + \cdots + g_c(\underline{Y})U^{c-b+1}.
\end{aligned}$$

The other assertion follows from this. □

Thirdly, we consider the case $1 < d < a$, $b \equiv d \pmod{a}$, that is, $b' \equiv 1 \pmod{a'}$.

Proposition 5.4.11. *If $1 < d < a$ and $b' = n_1 a' + 1$, then $\overline{G}(\mathfrak{m})$ is a complete intersection, and thus Gorenstein. In fact,*

$$\overline{G}(m) \cong K[X, \underline{Y}, Z]/(X^{a'}, Z^d - g_b(\underline{Y})).$$

Proof. Note that $b' \equiv 1 \pmod{a'}$. One can easily see that $n_k = kn_1$ for $1 \leq k \leq a' - 1$, and $n_{a'} = b'$. For any integer p ($1 \leq p \leq d - 1$), if $pa' + 1 \leq k \leq pa' + a' - 1$, then

$$n_k = \lfloor \frac{k(n_1 a' + 1)}{a'} \rfloor = kn_1 + \lfloor \frac{k}{a'} \rfloor = kn_1 + p = pb' + (k - pa')n_1.$$

Hence $(k, n_k) = p(a', b') + (k - pa')(1, n_1)$. Similarly, if $k = pa'$ ($2 \leq p \leq d - 1$), then $(k, n_k) = p(a', b')$. Therefore

$$\begin{aligned} \overline{\mathcal{R}'}(\mathfrak{m}) &= K[xt, y_1 t, \dots, y_m t, xt^{n_1}, x^{a'} t^{b'}, t^{-1}] \\ &\cong K[X, \underline{Y}, Z, U]/(X^{a'} - ZU, Z^d - g_b(\underline{Y}) - \dots - g_c(\underline{Y})U^{c-b}), \end{aligned}$$

where $X = xt^{n_1}$, $Y_i = y_i t$, $Z = x^{a'} t^{b'}$, and $U = t^{-1}$. Indeed,

$$X^{a'} = x^{a'} t^{n_1 a'} = (x^{a'} t^{b'}) (t^{-1})^{b' - n_1 a'} = (x^{a'} t^{b'}) (t^{-1}) = ZU.$$

and

$$\begin{aligned} Z^d &= (x^{a'} t^{b'})^d \\ &= x^{a'} t^b \\ &= g_b(y_1, \dots, y_m) t^b + \dots + g_c(y_1, \dots, y_m) t^c \cdot t^{b-c} \\ &= g_b(\underline{Y}) + \dots + g_c(\underline{Y}) U^{c-b}. \end{aligned}$$

The other assertion follows from this. □

In the rest of this section, we consider the case of $1 < d < a$ and $b' \equiv \delta \pmod{a'}$ for some δ ($2 \leq \delta \leq a' - 1$). Note that

$$\bar{\mathfrak{r}}(\mathfrak{m}) = n_{a-1} = \lfloor \frac{(a-1)b'}{a'} \rfloor = (a-1)n_1 + d\delta - 1.$$

In order to prove Theorem 5.4.8, it suffices to prove the following proposition.

Proposition 5.4.12. *In the above notation, if we put $k_1 = \lceil \frac{a'}{\delta} \rceil$ and $t_1 = (k_1 - 1)n_1 + 1$, then*

$$\ell_{t_1} + \ell_{r+1-t_1} = a + 1 (\neq a).$$

In particular, $\overline{G}(\mathfrak{m})$ is not Gorenstein.

Proof. We first prove the following claim.

Claim 1. $\lfloor \frac{k\delta}{a'} \rfloor = 0, \lceil \frac{k\delta}{a'} \rceil = 1$ for each k ($1 \leq k \leq k_1 - 1$).

By definition, $k \leq k_1 - 1 < \frac{a'}{\delta}$. Thus $0 < \frac{k\delta}{a'} < 1$. Hence $\lfloor \frac{k\delta}{a'} \rfloor = 0$ and $\lceil \frac{k\delta}{a'} \rceil = 1$.

Claim 2. $\lfloor \frac{k_1\delta}{a'} \rfloor = 1, \lceil \frac{k_1\delta}{a'} \rceil = 2$.

By definition, $k_1 - 1 < \frac{a'}{\delta} \leq k_1$. Since $(a', \delta) = 1$, $\frac{a'}{\delta}$ is not an integer. Hence $\frac{a'}{\delta} < k_1$, that is, $\frac{k_1\delta}{a'} > 1$. If $\frac{k_1\delta}{a'} \geq 2$, then one has

$$\frac{2a'}{\delta} - 1 \leq k_1 - 1 < \frac{a'}{\delta}.$$

This yields $a' < \delta$, which is a contradiction. Hence $1 < \frac{k_1\delta}{a'} < 2$. The claim follows from here.

Claim 3. $n_k = kn_1$ for every k ($1 \leq k \leq k_1 - 1$) and $n_{k_1} = k_1n_1 + 1$.

Indeed, for every k ($1 \leq k \leq k_1 - 1$), Claim 1 implies

$$n_k = \lfloor \frac{kb'}{a'} \rfloor = \lfloor \frac{k(n_1a' + \delta)}{a'} \rfloor = kn_1 + \lfloor \frac{k\delta}{a'} \rfloor = kn_1.$$

On the other hand, Claim 2 implies

$$n_{k_1} = \lfloor \frac{k_1b'}{a'} \rfloor = \lfloor \frac{k_1(n_1a' + \delta)}{a'} \rfloor = k_1n_1 + \lfloor \frac{k_1\delta}{a'} \rfloor = k_1n_1 + 1.$$

Claim 4. $n_{a-k} = (a-k)n_1 + d\delta - 1$ for each k ($1 \leq k \leq k_1 - 1$) and $n_{a-k_1} = (a-k_1)n_1 + d\delta - 2$.

For every k ($1 \leq k \leq k_1 - 1$), Claim 1 implies

$$n_{a-k} = \lfloor \frac{(a-k)(n_1a' + \delta)}{a'} \rfloor = (a-k)n_1 + d\delta - \lceil \frac{k\delta}{a'} \rceil = (a-k)n_1 + d\delta - 1.$$

On the other hand, Claim 2 implies

$$n_{a-k_1} = \lfloor \frac{(a-k_1)(n_1a' + \delta)}{a'} \rfloor = (a-k_1)n_1 + d\delta - \lceil \frac{k_1\delta}{a'} \rceil = (a-k_1)n_1 + d\delta - 2.$$

Claim 5. $\ell_{t_1} + \ell_{r+1-t_1} = a + 1$.

By Claim 3, $n_{k_1-1} + 1 = t_1 \leq n_{k_1}$. Hence $\ell_{t_1} = k_1$ by Proposition [5.4.7](#). By Claim 4, we have

$$n_{a-k_1} + 1 = (a-k_1)n_1 + d\delta - 1 = r + 1 - t_1$$

$$n_{a-k_1+1} = (a-k_1+1)n_1 + d\delta - 1 > r + 1 - t_1.$$

Hence $\ell_{r+1-t_1} = a - k_1 + 1$ and thus $\ell_{t_1} + \ell_{r+1-t_1} = (a - k_1 + 1) + k_1 = a + 1$. $\square$

Using Theorem 5.4.8, we can determine all Zariski-type hypersurfaces having maximal relative normal reduction number.

Proposition 5.4.13. *Let $A = S/(x^a - g(\underline{y}))$ be a Zariski-type hypersurface with $b = \text{ord}_{(\underline{y})}(g)$. Then*

- (a) $\bar{r}(\mathfrak{m}) = 1$ if and only if $(a, b) = (2, 2), (2, 3)$.
- (b) $\bar{r}(\mathfrak{m}) = 2$ if and only if $(a, b) = (2, 4), (2, 5), (3, 3), (3, 4)$.
- (c) $\text{nr}(\mathfrak{m})$ is maximal if and only if $(a, b) = (2, n), (3, 3), (3, 4), (3, 5)$, where $n \geq 2$ is any integer.

Proof. Note that $\bar{r}(\mathfrak{m}) = \lfloor \frac{(a-1)b}{a} \rfloor \geq a-1$ by Theorem 5.4.8.

(a) $\bar{r}(\mathfrak{m}) = 1$ implies $a = 2$ by the above remark. Thus one can easily see that $\bar{r}(\mathfrak{m}) = 1$ if $(a, b) = (2, 2), (2, 3)$.

(b) $\bar{r}(\mathfrak{m}) = 2$ implies $a = 2, 3$ by the above remark. When $a = 2$, $\bar{r}(\mathfrak{m}) = \lfloor \frac{b}{2} \rfloor = 2$ if and only if $b = 4, 5$. When $a = 3$, $\bar{r}(\mathfrak{m}) = \lfloor \frac{2b}{3} \rfloor = 2$ if and only if $b = 3, 4$.

(c) Suppose $r = \text{nr}(\mathfrak{m})$ is maximal. We may assume $r \geq 2$. By Theorem 5.2.2, we have $\ell(\overline{\mathfrak{m}^n}/Q\overline{\mathfrak{m}^{n-1}}) = 1$ for all $n = 2, \dots, r$. On the other hand, using Theorem 5.4.8, we can show that $\ell(\overline{\mathfrak{m}^n}/Q\overline{\mathfrak{m}^{n-1}}) = \max\{a - \lceil \frac{an}{b} \rceil, 0\}$ for all $n \geq 2$. Thus

$$a - 1 = \lceil \frac{2a}{b} \rceil \leq \lceil \frac{3a}{b} \rceil \leq \dots \leq \lceil \frac{ra}{b} \rceil \leq a - 1.$$

This means $\lceil \frac{2a}{b} \rceil = a - 1$. Then $a - 1 = \lceil \frac{2a}{b} \rceil \leq 2$ yields $a \leq 3$. By a similar argument as in the proof of (b), we obtain the required result. It is not so difficult to check that the converse is also true. $\square$

The following example gives a Zariski-type hypersurface such that $\text{nr}(\mathfrak{m})$ is maximal but $\overline{G}(\mathfrak{m})$ is not Gorenstein.

Example 5.4.14. Let $m \geq 1$ be an integer, and let $A = \mathbb{C}[x, y_1, \dots, y_m]/(x^3 + y_1^5 + \dots + y_m^5)$. Put $\mathfrak{m} = (x, y_1, \dots, y_m)$. Then

- (a) $A_{\mathfrak{m}}$ is a rational singularity if and only if $m \geq 4$.
- (b) $\overline{\mathfrak{m}^2} = \mathfrak{m}^2$, $\overline{\mathfrak{m}^3} = \mathfrak{m}^3 + (x^2)$, and $\overline{\mathfrak{m}^4} = (y_1, y_2, \dots, y_m)\overline{\mathfrak{m}^3}$.

(c) $\overline{\mathcal{R}'}(\mathfrak{m}) = K[xt, y_1t, y_2t, \dots, y_mt, x^2t^3, t^{-1}]$.

(d) $\overline{G}(\mathfrak{m})$ is a Cohen-Macaulay ring of $e_0(\overline{G}) = 3$ and CM type 2.

If we put $X = xt$, $Y_i = y_it$ ($i = 1, 2, \dots, m$) and $Z = x^2t^3$, and $U = t^{-1}$, then

$$\overline{\mathcal{R}'}(\mathfrak{m}) \cong K[X, \underline{Y}, Z, U]/(X^2 - ZU, XZ + \underline{Y}^5U, Z^2 + X\underline{Y}^5).$$

In particular,

$$\overline{G}(\mathfrak{m}) \cong \overline{\mathcal{R}'}(\mathfrak{m})/t^{-1}\overline{\mathcal{R}'}(\mathfrak{m}) \cong K[X, \underline{Y}, Z]/(X^2, XZ, Z^2 + X\underline{Y}^5),$$

where $\underline{Y}^5 = Y_1^5 + \dots + Y_m^5$.

Example 5.4.15. Let $A = K[[t^a, t^b]]$ be a semigroup ring generated by the positive integers $a \leq b$ such that $\gcd(a, b) = 1$. We know that $A \cong k[[x, y]]/(x^a - y^b)$ is a hypersurface. By Theorem 5.4.5 $\bar{r}_Q(\mathfrak{m}) = \lfloor \frac{(a-1)b}{a} \rfloor$. By Theorem 5.4.8, $\overline{G}(\mathfrak{m})$ is Gorenstein if and only if $b \equiv 1 \pmod{a}$.

The examples 5.3.3 and 5.3.4 can also be verified using Theorem 5.4.8.

Chapter 6

Summary and Conclusion

In a nutshell, Chapters 2 and 3 are focused mainly on Waring decomposition of forms. We described the structure of minimal apolar sets of reducible forms $(x_1^2 + \cdots + x_m^2)(y_1^2 + \cdots + y_n^2)$ for $m = 1, 2$. For $m, n \geq 3$, only a bound on the Waring rank is known. Even in a specific case where $m = n = 3$, the Waring rank is unknown. It would be nice to explore more on the rank and understand the structure of minimal apolar sets for $m \geq 3$.

Further, we think that our techniques in Chapter 2 may be useful in understanding the structure of minimal apolar sets of degree 4 forms if the rank is known in general. Although this thesis does not state results for forms not of the type $(x_1^2 + \cdots + x_m^2)(y_1^2 + \cdots + y_n^2)$, it is an interesting direction to explore.

In Chapter 3, we computed the Waring rank of quasi-monomial complete intersection. This is a special case of Question C. In general, it is not so clear when is a homogeneous ideal of the form $(X_0^{a_0} + G_0, X_1^{a_1} + G_1, \dots, X_n^{a_n} + G_n)$ where $\deg G_i = a_i$ and $\deg_{X_i} G_i < a_i$ for $0 \leq i \leq n$ is a complete intersection. Further, it would be nice to characterize the homogeneous polynomials whose annihilator is a quasi-monomial complete intersection. By the results of [21, 67] we know which binomials are subsets of such class of forms.

We note that, in general computing the Waring rank of F such that its annihilator ideal $F^\perp$ is a complete intersection remains open. Moreover, the computation of the Waring rank of binomials whose annihilator is not a complete intersection, even in three variables, has not yet been solved.

In addition, we have not explored other notions of rank. For instance, the border rank and the cactus rank of forms that we have worked on in both chapters. It would be worth understanding if there is any gap between the Waring rank and border rank of such forms.

Chapter 4 opens a whole new direction for characterizing I -Gorenstein sequences. We have conjectures 4.3.3 and 4.3.6 on uniqueness of symmetric decomposition. Our investigation of small socle degree cases support conjectures, but this is far from having a complete solution.

In Chapter 5 our computation of the Hilbert series uses the hypothesis of $\text{depth} G(\mathcal{F}) \geq d - 1$ where d is the dimension of the ring. We do not have any efficient tools to tackle the problem if we drop the condition on the depth in general. Moreover, it would be nice to have a characterization for Gorensteinness if we assume $\text{depth} G(\mathcal{F}) = d - 1$ in Theorem 5.2.6 instead of assuming $G(\mathcal{F})$ is Cohen-Macaulay.

Appendix A

Apolarity Theory

A.1 Apolarity action

Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the standard graded polynomial ring and let $T = \mathbb{C}[X_0, \dots, X_n]$ be the ring of differential operators on S . The apolarity action of T on S is defined on monomials by

$$X_0^{a_0} \cdots X_n^{a_n} \circ (x_0^{b_0} \cdots x_n^{b_n}) = \frac{\partial^{a_0}}{\partial x_0^{a_0}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} (x_0^{b_0} \cdots x_n^{b_n}),$$

and then linearly extended on S .

Definition A.1. With respect to the action defined above, given a polynomial $F \in S$, we define an ideal

$$F^\perp = \{g \in T \mid g \circ F = 0\} \subseteq T.$$

Proposition A.2 ([39], Lemma 1.15). Let $L \in S_1$ be a linear form given by $L = \sum_{i=0}^n a_i x_i$, and $g \in T_\delta$ be such that $\delta \leq d$. Then

$$g \circ L^d = \frac{d!}{(d-\delta)!} g(a_0, a_1, \dots, a_n) L^{d-\delta}.$$

Proof. To prove this result, it is enough to assume g is a monomial. First we prove for $g = X_0$.

Consider

$$\begin{aligned}
X_0 \circ F &= X_0 \circ L^d \\
&= X_0 \circ \left(\sum_{i=0}^n a_i x_i \right)^d \\
&= X_0 \circ \left(a_0 x_0 + \left(\sum_{i=1}^n a_i x_i \right) \right)^d \\
&= X_0 \circ \left((a_0 x_0)^d + \binom{d}{1} (a_0 x_0)^{d-1} \left(\sum_{i=1}^n a_i x_i \right) + \cdots + \left(\sum_{i=1}^n a_i x_i \right)^d \right) \\
&= d a_0 (a_0 x_0)^{d-1} + a_0 (d-1) \binom{d}{1} (a_0 x_0)^{d-2} \left(\sum_{i=1}^n a_i x_i \right) + \cdots + a_0 \binom{d}{d-1} \left(\sum_{i=1}^n a_i x_i \right)^{d-1} \\
&= d a_0 (a_0 x_0)^{d-1} + d a_0 \binom{d-1}{1} (a_0 x_0)^{d-2} \left(\sum_{i=1}^n a_i x_i \right) + \cdots \\
&\quad + d a_0 \binom{d-1}{1} (a_0 x_0) \left(\sum_{i=1}^n a_i x_i \right)^{d-2} + d a_0 \left(\sum_{i=1}^n a_i x_i \right)^{d-1} \\
&= \frac{d!}{(d-1)!} a_0 L^{d-1}.
\end{aligned}$$

Let $b_0 \geq 1$, by induction on b_0 , we obtain that

$$X_0^{b_0} \circ L^d = \frac{d!}{(d-b_0)!} a_0^{b_0} L^{d-b_0}. \quad (\text{A.1.1})$$

Let $g = X_0^{b_0} X_1^{b_1} \cdots X_n^{b_n}$ be such that $\sum_{i=0}^n b_i = \delta \leq d$. Consider

$$\begin{aligned}
(X_0^{b_0} X_1^{b_1} \cdots X_n^{b_n}) \circ F &= (X_0^{b_0} X_1^{b_1} \cdots X_n^{b_n}) \circ L^d \\
&= (X_0^{b_0} X_1^{b_1} \cdots X_{n-1}^{b_{n-1}} X_n^{b_n}) \circ L^d \\
&= (X_0^{b_0} X_1^{b_1} \cdots X_{n-1}^{b_{n-1}}) \circ (X_n^{b_n} \circ L^d) \\
&= (X_0^{b_0} X_1^{b_1} \cdots X_{n-1}^{b_{n-1}}) \circ \left(\frac{d!}{(d-b_n)!} a_n^{b_n} L^{d-b_n} \right) \quad (\text{by Equation (A.1.1)}) \\
&= \frac{d!}{(d-b_n)!} \frac{(d-b_n)!}{(d-b_n-b_{n-1})!} \cdots \\
&\quad \frac{(d-b_n-\cdots-b_1)!}{(d-b_n-\cdots-b_1-b_0)!} (a_0^{b_0} \cdots a_n^{b_n}) L^{(d-b_n-\cdots-b_1-b_0)} \\
&= \frac{d!}{(d-\delta)!} (a_0^{b_0} \cdots a_n^{b_n}) L^{d-\delta}.
\end{aligned}$$

Therefore, we have proved the result. □

The following is a standard property of a non-zerodivisor in the quotient ring of an ideal of points, included here for the sake of completeness.

Lemma A.3. *Let $\mathbb{X} = \{P_1, \dots, P_l\}$ be a set of l points in $\mathbb{P}^n$. Then g is a non-zerodivisor in $T/I(\mathbb{X})$ if and only if $g(P_i) \neq 0$ for all $1 \leq i \leq l$.*

Proof. Let g be a non-zerodivisor in $T/I(\mathbb{X})$. This implies $g \notin I(\mathbb{X})$. Suppose there exists P_m such that $g(P_m) = 0$ for some $1 \leq m \leq l$. Let $\mathcal{P}_i$ denote the prime ideal corresponding to the point P_i for $1 \leq i \leq l$. By rearranging if required, assume that $g(P_i) \neq 0$ for $1 \leq i \leq e$ and $g(P_i) = 0$ for $e+1 \leq i \leq l$. Since $\mathcal{P}_i \not\subset \mathcal{P}_j$ for any i, j , we can choose $h \in \mathcal{P}_1 \cap \dots \cap \mathcal{P}_e \setminus \mathcal{P}_{e+1}$. Then $g \neq 0$ and $h \neq 0$ in $T/I(\mathbb{X})$, and $gh = 0$ in $T/I(\mathbb{X})$. This is a contradiction to our assumption that g is a non-zerodivisor.

Conversely, assume $g(P_i) \neq 0$ for all $1 \leq i \leq l$. Let $h \in T/I(\mathbb{X})$ such that $gh \in I(\mathbb{X})$. This implies, for all $1 \leq i \leq l$,

$$\begin{aligned} gh(P_i) &= 0 \\ g(P_i)h(P_i) &= 0 \\ h(P_i) &= 0 \quad (\text{because } g(P_i) \neq 0 \text{ for } 1 \leq i \leq l) \end{aligned}$$

Hence $h \in I(\mathbb{X})$. Therefore g is a non-zerodivisor. Hence proved. □

Appendix B

Gröbner basis computation

In this chapter, we present some computations in detail that we need in Chapter 3. Firstly, we recall some useful results in the theory of Gröbner bases and elimination theory. We use these results to compute the Gröbner bases of some ideals.

Let $T = \mathbb{C}[X_0, \dots, X_n]$ be the standard graded polynomial ring with the *lex ordering*, i.e., for $\alpha, \beta \in \mathbb{Z}_{n \geq 0}$, $\alpha >_{\text{lex}} \beta$ if the left most non-zero entry of the vector difference $\alpha - \beta \in \mathbb{Z}$ is positive.

Theorem B.1 ([20], Theorem 4.3.11). *Let I, J be ideals in T . If t is a new variable and $tI + (1 - t)J$ is considered as an ideal in $T[t]$, then*

$$I \cap J = (tI + (1 - t)J) \cap T.$$

Note that, if we compute a Gröbner basis GB of the ideal $tI + (1 - t)J$ using the lex ordering in which t is greater than X_i for all i , then the terms in GB not involving t are a Gröbner basis of $I \cap J$. The following theorem shows that we can use this information to compute a colon ideal.

Theorem B.2 ([20], Theorem 4.4.14). *Let $I \subset T$ be an ideal and $g \in T$. If $\{h_1, h_2, \dots, h_p\}$ is a generating set of $I \cap (g)$, then $\{h_1/g, h_2/g, \dots, h_p/g\}$ is a generating set of $I : (g)$.*

Next, we compute $F^\perp : (X_i)$, $1 \leq i \leq n$, and $F^\perp : (h)$ when $X_0^{a_0} + G = h^m g$ for some forms $h, g \in T$ with $\deg h > 0$ and $m \geq 2$.

Let $T[t] = \mathbb{C}[t, X_0, X_1, \dots, X_n]$ be the standard graded polynomial ring with the *lex ordering* $t > X_0 > \dots > X_n$.

Proposition B.3. *Let $I = (X_0^{a_0} + G, X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n}) \subset T$ be an ideal.*

1. If $J = (X_i) \subset T$ for some $1 \leq i \leq n$, then

$$I \cap J = (X_i(X_0^{a_0} + G), X_iX_1^{a_1}, X_iX_2^{a_2}, \dots, X_i^{a_i}, \dots, X_iX_n^{a_n}).$$

2. Suppose $X_0^{a_0} + G = h^m g$ for some forms $h, g \in T$ with $\deg h = k > 0$ and $m \geq 2$. If $J = (h)$, then

$$I \cap J = (h^m g, hX_1^{a_1}, hX_2^{a_2}, \dots, hX_n^{a_n}).$$

Proof. We start by proving (1). We show that a Gröbner basis of $tI + (1-t)J \subset T[t]$ is,

$$GB = \{tX_0^{a_0} + tG, tX_1^{a_1}, tX_2^{a_2}, tX_3^{a_3}, \dots, tX_n^{a_n}, (t-1)X_i, X_i(X_0^{a_0} + G), X_iX_1^{a_1}, X_iX_2^{a_2}, \dots, X_i^{a_i}, \dots, X_iX_n^{a_n}\}.$$

Thus the conclusion follows by Theorem [B.1](#).

To show that GB is a Gröbner basis, we show that $S(f_i, f_j) = 0 \pmod{GB}$ for all distinct $f_i, f_j \in GB$. Note that if f_i, f_j are monomials, then $S(f_i, f_j) = 0$. Also, if the leading terms of f_i, f_j are coprime, then $S(f_i, f_j) = 0 \pmod{GB}$. We compute $S(f_i, f_j)$ for the remaining pairs. We have

$$\begin{aligned} S(tX_0^{a_0} + tG, tX_i - X_i) &= \frac{tX_0^{a_0}X_i}{tX_0^{a_0}}(tX_0^{a_0} + tG) - \frac{tX_0^{a_0}X_i}{tX_i}(tX_i - X_i) \\ &= tX_iG + X_iX_0^{a_0} \\ &= X_i(X_0^{a_0} + G) - GX_i(1-t) = 0 \pmod{GB}. \end{aligned}$$

$$\begin{aligned} S(X_i(X_0^{a_0} + G), tX_i - X_i) &= \frac{tX_iX_0^{a_0}}{X_iX_0^{a_0}}(X_iX_0^{a_0} + X_iG) - \frac{tX_iX_0^{a_0}}{tX_i}(tX_i - X_i) \\ &= tX_iG + X_0^{a_0}X_i \\ &= tX_iG - X_iG + X_iG + X_iX_0^{a_0} \\ &= (t-1)X_iG + X_i(X_0^{a_0} + G) = 0 \pmod{GB}. \end{aligned}$$

$$\begin{aligned} S(tX_j^{a_j}, tX_i - X_i) &= \frac{tX_j^{a_j}X_i}{tX_j^{a_j}}(tX_j^{a_j}) - \frac{tX_j^{a_j}X_i}{tX_i}(tX_i - X_i) \\ &= X_j^{a_j}X_i = 0 \pmod{GB} \quad \text{for all } 1 \leq j \leq n, i \neq j. \end{aligned}$$

$$\begin{aligned} S(X_i(X_0^{a_0} + G), X_iX_j^{a_j}) &= \frac{X_iX_0^{a_0}X_j^{a_j}}{X_iX_0^{a_0}}(X_iX_0^{a_0} + X_iG) - \frac{X_iX_0^{a_0}X_j^{a_j}}{X_iX_j^{a_j}}(X_iX_j^{a_j}) \\ &= X_iX_j^{a_j}G = 0 \pmod{GB} \quad \text{for all } 1 \leq j \leq n, i \neq j. \end{aligned}$$

Hence (1) proved.

To prove (2) we perform similar computations as above and prove that

$$GB_1 = \{th^m g, tX_1^{a_1}, tX_2^{a_2}, \dots, tX_n^{a_n}, (t-1)h, h^m g, hX_1^{a_1}, hX_2^{a_2}, \dots, hX_n^{a_n}\}$$

is a Gröbner basis of $tI + (1-t)J \subset T[t]$. Then the conclusion follows by Theorem [B.1](#).

To prove that GB_1 is a Gröbner basis, it is enough to compute the S-polynomials and their remainder modulo GB_1 . Consider

$$\begin{aligned} S(th^m g, tX_j^{a_j}) &= \frac{tX_0^{a_0} X_j^{a_j}}{tX_0^{a_0}}(th^m g) - \frac{tX_0^{a_0} X_j^{a_j}}{tX_j^{a_j}}(tX_j^{a_j}) \\ &= tX_j^{a_j} h^m g - tX_0^{a_0} X_j^{a_j} \\ &= tX_j^{a_j} (h^m g - X_0^{a_0}) = 0(\text{mod } GB_1). \end{aligned}$$

$$\begin{aligned} S(th^m g, th - h) &= \frac{tX_0^{a_0}}{tX_0^{a_0}}(th^m g) - \frac{tX_0^{a_0}}{tX_0^k}(th - h) \\ &= th^m g - thX_0^{a_0-k} + hX_0^{a_0-k} \\ &= th(h^{m-1}g - X_0^{a_0-k}) + hX_0^{a_0-k} \\ &= th(h^{m-1}g - X_0^{a_0-k}) - h(h^{m-1}g - X_0^{a_0-k}) + h(h^{m-1}g - X_0^{a_0-k}) + hX_0^{a_0-k} \\ &= (t-1)h(h^{m-1}g - X_0^{a_0-k}) + h^m g = 0(\text{mod } GB_1). \end{aligned}$$

$$S(th^m g, h^m g) = \frac{tX_0^{a_0}}{tX_0^{a_0}}(th^m g) - \frac{tX_0^{a_0}}{X_0^{a_0}}(h^m g) = 0.$$

$$\begin{aligned} S(th^m g, hX_j^{a_j}) &= \frac{tX_0^{a_0} X_j^{a_j}}{tX_0^{a_0}}(th^m g) - \frac{tX_0^{a_0} X_j^{a_j}}{X_0^k X_j^{a_j}}(hX_j^{a_j}) \\ &= tX_j^{a_j} h^m g - thX_0^{a_0-k} X_j^{a_j} \\ &= tX_j^{a_j} (h^m g - hX_0^{a_0-k}) = 0(\text{mod } GB_1). \end{aligned}$$

$$\begin{aligned} S(tX_j^{a_j}, th - h) &= \frac{tX_j^{a_j} X_0^k}{tX_j^{a_j}}(tX_j^{a_j}) - \frac{tX_j^{a_j} X_0^k}{tX_0^k}(th - h) \\ &= tX_j^{a_j} X_0^k - tX_j^{a_j} h + hX_j^{a_j} \\ &= tX_j^{a_j} (X_0^k - h) + hX_j^{a_j} = 0(\text{mod } GB_1). \end{aligned}$$

$$\begin{aligned} S(tX_j^{a_j}, hX_j^{a_j}) &= \frac{tX_j^{a_j} X_0^k}{tX_j^{a_j}}(tX_j^{a_j}) - \frac{tX_j^{a_j} X_0^k}{X_0^k X_j^{a_j}}(hX_j^{a_j}) \\ &= tX_j^{a_j} (X_0^k - h) = 0(\text{mod } GB_1). \end{aligned}$$

$$\begin{aligned}
S(th - h, hX_j^{a_j}) &= \frac{tX_0^k X_j^{a_j}}{tX_0^k} (th - h) - \frac{tX_0^k X_j^{a_j}}{X_0^k X_j^{a_j}} (hX_j^{a_j}) \\
&= -hX_j^{a_j} = 0 \pmod{\text{GB}_1}.
\end{aligned}$$

$$\begin{aligned}
S(th - h, h^m g) &= \frac{tX_0^{a_0}}{tX_0^k} (th - h) - \frac{tX_0^{a_0}}{X_0^{a_0}} (h^m g) \\
&= tX_0^{a_0-k} h - X_0^{a_0-k} h - th^m g \\
&= th(X_0^{a_0-k} - h^{m-1} g) - hX_0^{a_0-k} \\
&= th(X_0^{a_0-k} - h^{m-1} g) - h(X_0^{a_0-k} - h^{m-1} g) + h(X_0^{a_0-k} - h^{m-1} g) - hX_0^{a_0-k} \\
&= (t-1)h(X_0^{a_0-k} - h^{m-1} g) - h^m g = 0 \pmod{\text{GB}_1}.
\end{aligned}$$

$$\begin{aligned}
S(h^m g, hX_j^{a_j}) &= \frac{X_0^{a_0} X_j^{a_j}}{X_0^{a_0}} (h^m g) - \frac{X_0^{a_0} X_j^{a_j}}{X_0^k X_j^{a_j}} (hX_j^{a_j}) \\
&= X_j^{a_j} h^m g - X_0^{a_0-k} hX_j^{a_j} \\
&= hX_j^{a_j} (h^{m-1} g - X_0^{a_0-k}) = 0 \pmod{\text{GB}_1}.
\end{aligned}$$

$$\begin{aligned}
S(h^m g, hX_j^{a_j}) &= \frac{X_0^{a_0} X_j^{a_j}}{X_0^{a_0}} (h^m g) - \frac{X_0^{a_0} X_j^{a_j}}{X_0^k X_j^{a_j}} (hX_j^{a_j}) \\
&= X_j^{a_j} h^m g - X_0^{a_0-k} hX_j^{a_j} \\
&= hX_j^{a_j} (h^{m-1} g - X_0^{a_0-k}) = 0 \pmod{\text{GB}_1} \text{ for } j = 1, \dots, n.
\end{aligned}$$

Hence the proof is complete. □

We compute some colon ideals required for the Waring rank computation.

Lemma B.4. Let $F^\perp = (P = \sum_{i=0}^v X_0^{ib_0} X_1^{(v-i)b_0}, Q = \sum_{i=0}^{v-1} X_0^{a_0+1-(v-1-i)b_0} X_1^{a_1+1-ib_0}, X_2^{a_2}, \dots, X_n^{a_n})$ where v is the least integer such that $a_0 + 1 \leq vb_0$ and $a_1 + 1 \leq vb_0$.

1. If $I = (X_2)$, then $F^\perp : I = (P, Q, X_2^{a_2-1}, X_3^{a_3}, \dots, X_n^{a_n})$.
2. If $I = (X_1)$, then $F^\perp : I = (P, Q/X_1, X_2^{a_2}, \dots, X_n^{a_n})$.
3. If $I = (X_0, X_1)$, then $F^\perp : I = (P, Q, X_2^{a_2}, \dots, X_n^{a_n})$.

Proof. To prove part (1) we note that by performing computations similar to Proposition [B.3](#),

we obtain a Gröbner basis of the ideal $tF^\perp + (1-t)I \subset T[t]$ given by

$$GB_1 = \{tP, tQ, tX_2^{a_2}, \dots, tX_n^{a_n}, (t-1)X_2, tX_1^{a_1+1+b_0}, X_2P, X_2Q, X_2^{a_2}, X_2X_3^{a_3}, \dots, X_2X_n^{a_n}\}.$$

By Theorem [B.2](#) we obtain the result (1).

We now prove (2). We show that a Gröbner basis of the ideal $tF^\perp + (1-t)I \subset T[t]$ is

$$GB_2 = \{tP, tQ, tX_2^{a_2}, \dots, tX_n^{a_n}, (t-1)X_1, X_1^{a_1+1+b_0}, X_1P, Q, X_1X_2^{a_2}, \dots, X_1X_n^{a_n}\}.$$

To show that GB_2 is a Gröbner basis, we show that $S(f_i, f_j) = 0 \pmod{GB_2}$ for all distinct $f_i, f_j \in GB_2$. Note that if f_i, f_j are monomials, then $S(f_i, f_j) = 0$. Also, if the leading terms of f_i, f_j are coprime, then $S(f_i, f_j) = 0 \pmod{GB_2}$. We compute $S(f_i, f_j)$ for the remaining pairs. We have

$$\begin{aligned} S(tP, tQ) &= X_1^{a_1+1-(v-1)b_0}tP - X_0^{vb_0-(a_0+1)}tQ \\ &= tX_1^{a_1+1+b_0} \\ &= (t-1)X_1^{a_1+b_0+1} + X_1^{a_1+1+b_0} = 0 \pmod{GB_2}. \end{aligned}$$

$$\begin{aligned} S(tP, tX_j^{a_j}) &= X_j^{a_j}tP - X_0^{vb_0}tX_j^{a_j} \\ &= tX_j^{a_j} \left(\sum_{i=0}^{v-1} X_0^{ib_0} X_1^{(v-i)b_0} \right) = 0 \pmod{GB_2}. \end{aligned}$$

$$\begin{aligned} S(tP, (t-1)X_1) &= X_1tP - X_0^{vb_0}(t-1)X_1 \\ &= (t-1)X_1 \left(\sum_{i=0}^{v-1} X_0^{ib_0} X_1^{(v-i)b_0} \right) + X_1P = 0 \pmod{GB_2}. \end{aligned}$$

$$\begin{aligned} S(tQ, tX_j^{a_j}) &= X_j^{a_j}tQ - X_0^{a_0+1}X_1^{a_1+1-(v-1)b_0}tX_j^{a_j} \\ &= tX_j^{a_j} \left(\sum_{i=0}^{v-2} X_0^{a_0+1-(v-1-i)b_0} X_1^{a_1+1-ib_0} \right) = 0 \pmod{GB_2}. \end{aligned}$$

$$\begin{aligned} S(tQ, (t-1)X_1) &= tQ - X_0^{a_0+1}(t-1)X_1^{a_1+1-(v-1)b_0} \\ &= (t-1)X_1 \left(\sum_{i=0}^{v-2} X_0^{a_0+1-(v-1-i)b_0} X_1^{a_1+1-b_0-1} \right) + Q = 0 \pmod{GB_2}. \end{aligned}$$

$$S(tQ, Q) = tQ - tQ = 0 \pmod{GB_2}.$$

$$\begin{aligned} S(tQ, X_1X_j^{a_j}) &= X_j^{a_j}tQ - tX_0^{a_0+1}X_j^{a_j}X_1^{a_1+1-(v-1)b_0} \\ &= tX_j^{a_j}(Q - X_0^{a_0+1}X_1^{a_1+1-(v-1)b_0}) = 0 \pmod{GB_2}. \end{aligned}$$

$$S(tQ, X_1^{a_1+1+b_0}) = X_1^{vb_0}tQ - tX_0^{a_0+1}X_1^{a_1+1+b_0}$$

$$= X_1^{a_1+1+b_0} \left(\sum_{i=0}^{v-2} X_0^{a_0+1-(v-1-i)b_0} X_1^{(v-1-i)b_0} \right) = 0 \mod GB_2.$$

$$\begin{aligned} S(tX_j^{a_j}, (t-1)X_1) &= X_1 tX_j^{a_j} - X_j^{a_j} (t-1)X_1 \\ &= X_1 X_j^{a_j} = 0 \mod GB_2. \end{aligned}$$

$$\begin{aligned} S((t-1)X_1, X_1^{a_1+1+b_0}) &= X_1^{a_1+b_0} (t-1)X_1 - tX_1^{a_1+1+b_0} \\ &= -X_1^{a_1+1+b_0} = 0 \mod GB_2. \end{aligned}$$

$$\begin{aligned} S(X_1 P, Q) &= X_1^{a_1+1-(v-1)b_0} P - X_0^{vb_0-(a_0+1)} Q \\ &= X_1^{a_1+1+b_0} = 0 \mod GB_2. \end{aligned}$$

$$\begin{aligned} S(X_1 P, X_1 X_j^{a_j}) &= X_j^{a_j} X_1 P - X_0^{vb_0} X_1 X_j^{a_j} \\ &= X_1 X_j^{a_j} \sum_{i=0}^{v-1} X_0^{ib_0} X_1^{(v-i)b_0} = 0 \mod GB_2. \end{aligned}$$

$$\begin{aligned} S(Q, X_1 X_j^{a_j}) &= X_j^{a_j} Q - X_0^{a_0+1} X_1^{a_1+1-b_0} \\ &= X_1 X_j^{a_j} \sum_{i=0}^{v-2} X_0^{a_0+1-(v-1-i)b_0} X_1^{a_1-i b_0} = 0 \mod GB_2. \end{aligned}$$

Therefore

$$F^\perp \cap I = (X_1^{a_1+1+b_0}, X_1 P, Q, X_1 X_2^{a_2}, \dots, X_1 X_n^{a_n}).$$

By Theorem [B.2](#) we obtain

$$F^\perp : I = (X_1^{a_1+b_0}, P, Q/X_1, X_2^{a_2}, \dots, X_n^{a_n}).$$

Since

$$X_1^{a_1+b_0} = X_1^{a_1+b_0-vb_0} P - X_0^{vb_0-(a_0+1)} Q/X_1$$

we get

$$F^\perp : I = (P, Q/X_1, X_2^{a_2}, \dots, X_n^{a_n}).$$

To prove part (3) we first obtain $F^\perp : (X_0)$. By performing computations similar to the Case (2) above, we obtain that a Gröbner basis of the ideal $tF^\perp + (1-t)(X_0) \subset T[t]$ is

$$\begin{aligned} GB_2 = \{ & tP, tQ, tX_2^{a_2}, \dots, tX_n^{a_n}, (t-1)X_0, tX_1^{vb_0} + \sum_{i=1}^v X_0^{ib_0} X_1^{(v-i)b_0}, X_0 X_1^{a_1+1+b_0}, X_0^{a_0+b_0+1} \\ & X_0 P, Q, X_0 X_2^{a_2}, X_0 X_3^{a_3}, \dots, X_0 X_n^{a_n} \}. \end{aligned}$$

By Theorem [B.2](#) we obtain

$$F^\perp : (X_0) = (X_1^{a_1+1+b_0}, X_0^{a_0+b_0}, P, Q/X_0, X_2^{a_2}, X_3^{a_3}, \dots, X_n^{a_n})$$

Moreover,

$$\begin{aligned} X_1^{a_1+1+b_0} &= X_1^{a_1+1+b_0-vb_0} P - X_0^{vb_0-a_0} Q/X_0, \\ X_0^{a_0+b_0} &= X_0^{a_0+b_0-vb_0} P - X_1^{vb_0-(a_1+1)} Q/X_0. \end{aligned}$$

Hence

$$F^\perp : (X_0) = (P, Q/X_0, X_2^{a_2}, \dots, X_n^{a_n}).$$

Therefore we have

$$\begin{aligned} F^\perp : (X_0, X_1) &= (F^\perp : (X_0)) \cap (F^\perp : (X_1)) \\ &= (P, Q, X_2^{a_2}, \dots, X_n^{a_n}). \end{aligned}$$

Hence the proof is complete. □

Appendix C

Artinian Gorenstein algebras and bilinear map

In this chapter, we recall some properties of AG k -algebras and bilinear maps. Some of these properties are well-known, but we have added details of some results for the sake of completeness.

C.1 AG k -algebras

Let $(R, \mathcal{M})$ be a local ring over a field $k \cong R/\mathcal{M}$. In this subsection we recall that there is a one-many correspondence between AG quotients of R and k -linear maps from $R \rightarrow k$ satisfying certain conditions. This map induces a non-degenerate bilinear map from $A \times A \rightarrow k$.

Proposition C.1. *Let $(R, \mathcal{M})$ be a local ring over a field k . There is a one-many correspondence between the following sets:*

$$\left\{ \begin{array}{l} \text{AG quotients } A \text{ of } R \\ \text{of socle degree } s \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} k\text{-linear map } \varphi : R \rightarrow k \text{ such that} \\ \varphi|_{\mathcal{M}^{s+1}} = 0, \varphi|_{\mathcal{M}^s} \neq 0. \end{array} \right\}$$

Here $A = R/J$ with $J = \{h \in R \mid \varphi(Rh) = 0\}$.

Proof. Refer to [38, Lemma 1.1]. □

C.2 Properties of bilinear maps

Let k be a field and V, W be finite-dimensional k -vector spaces. Recall that a map $\varphi : V \times W \rightarrow k$ is said to be bilinear if it satisfies the following properties: for any $x, y \in V, z, w \in W$ and $\alpha \in k$,

- (i) $\varphi(\alpha(x + y), z) = \alpha \varphi(x, z) + \alpha \varphi(y, z)$;
- (ii) $\varphi(x, \alpha(z + w)) = \alpha \varphi(x, z) + \alpha \varphi(x, w)$.

Let $\varphi : V \times W \rightarrow k$ be a bilinear map. For a subset S of V , we define a subset $S^\perp$ of W as

$$S^\perp := \{w \in W \mid \varphi(v, w) = 0 \text{ for every } v \in S\}.$$

Similarly, if T is a subset of W , then a subset $\text{Ann}_V(T)$ of V is

$$\text{Ann}_V(T) := \{v \in V \mid \varphi(v, t) = 0 \text{ for all } t \in T\}.$$

Note that both $\text{Ann}_V(T)$ and $S^\perp$ are subspaces of V and W , respectively. For subsets S, O of V recall that

$$S + O := \{x + y : x \in S \text{ and } y \in O\}.$$

Proposition C.1. *Let $\varphi : V \times W \rightarrow k$ be a bilinear map and $S, O \subseteq V$, and $T, U \subseteq W$ are subsets. Then*

- (i) *If $T \subseteq U$, then $\text{Ann}_V(U) \subseteq \text{Ann}_V(T)$.*
- (ii) *If $S \subseteq O$, then $O^\perp \subseteq S^\perp$.*
- (iii) $\text{Ann}_V(T + U) = \text{Ann}_V(T) \cap \text{Ann}_V(U)$.
- (iv) $(S + O)^\perp = S^\perp \cap O^\perp$.

Proof. Easy to check. □

Throughout, for any k -vector space V , we set $\check{V} := \text{Hom}(V, k)$. For a subset S of V , let

$$(0 : {}_V S) := \{f \in \check{V} : f(t) = 0 \text{ for all } t \in S\}.$$

Note that $(0 : {}_V S)$ is a subspace of $\check{V}$. We denote the k -vector space dimension of V by $\dim V$.

Proposition C.2. *Let V, W be finite dimensional vector spaces of the same dimension, and $S \subseteq V, T \subseteq W$ are subspaces. Let $\varphi : V \times W \rightarrow k$ be a bilinear map.*

(i) *If the k -linear map $\varphi' : V \rightarrow \text{Hom}(W, k)$ induced by φ is injective, then*

$$(0 : {}_W T) \cong \text{Ann}_V(T).$$

Moreover, $\dim \text{Ann}_V(T) = \dim W - \dim T$.

(ii) *If the k -linear map $\varphi'' : W \rightarrow \text{Hom}(V, k)$ induced by φ is injective, then*

$$(0 : {}_V S) \cong S^\perp.$$

Moreover, $\dim S^\perp = \dim V - \dim S$.

(iii) *If φ is non-degenerate, then*

$$\text{Ann}_V(S^\perp) = S, (\text{Ann}_V T)^\perp = T.$$

(iv) *Assume φ is non-degenerate. Let S, O be subspaces of V and T, U be subspaces of W .*

Then

$$(S \cap O)^\perp = S^\perp + O^\perp, \text{Ann}_V(T \cap U) = \text{Ann}_V(T) + \text{Ann}_V(U).$$

Proof. (i) Since $\dim V = \dim W$, $\dim V = \dim W^\vee$. Therefore, if

$$\varphi' : V \rightarrow \text{Hom}(W, k) = W^\vee$$

induced by φ is injective, then φ' is surjective as well. Hence φ' is an isomorphism.

Let ψ be the restriction map of φ' to $\text{Ann}_V(T)$, that is $\psi : \text{Ann}_V(T) \rightarrow W^\vee$ is defined as $\psi(v) = \phi'(v)$. Note that for $v \in \text{Ann}_V(T)$,

$$\phi'(v)(t) = \phi(v, t) = 0 \text{ for all } t \in T.$$

Hence $\psi(v) = \phi'(v) \in (0 : {}_W T)$. Therefore $\text{im}(\psi) \subseteq (0 : {}_W T)$. Clearly, as ϕ' is injective, ψ is injective. Also, for any $f \in (0 : {}_W T)$, surjectivity of ϕ' implies that there exists $v \in V$ such that $\phi'(v) = f$. Then $v \in \text{Ann}_V(T)$ because

$$\phi(v, t) = \phi'(v)(t) = f(t) = 0 \text{ for all } t \in T.$$

Thus $\psi(v) = \phi'(v) = f$. Hence $\text{im}(\psi) = (0 : {}_W T)$. Thus we get $\text{Ann}_V(T) \cong (0 : {}_W T)$.

In order to compute the dimension of $\text{Ann}_V(T)$, first observe that $(0: {}_W T) \simeq (W/T)^\vee$. Hence

$$\dim \text{Ann}_V(T) = \dim(0: {}_W T) = \dim(W/T)^\vee = \dim W/T = \dim W - \dim T.$$

(ii) Proof is similar to (i).

(iii) We have

$$\begin{aligned} \dim \text{Ann}_V(S^\perp) &= \dim W - \dim S^\perp && \text{(from (i))} \\ &= \dim W - (\dim V - \dim S) && \text{(from (ii))} \\ &= \dim S && \text{(since } \dim W = \dim V \text{).} \end{aligned}$$

Since $S \subseteq \text{Ann}_V(S^\perp)$, we obtain that $\text{Ann}_V(S^\perp) = S$.

Similarly, $(\text{Ann}_V T)^\perp = T$ is proved.

(iv) We have

$$\begin{aligned} \text{Ann}_V(S^\perp + O^\perp) &= \text{Ann}_V(S^\perp) \cap \text{Ann}_V(O^\perp) && \text{(by Proposition C.1(iii))} \\ &= S \cap O && \text{(by part (iii)).} \end{aligned}$$

Therefore we obtain

$$\begin{aligned} (S \cap O)^\perp &= (\text{Ann}_V(S^\perp + O^\perp))^\perp \\ &= S^\perp + O^\perp && \text{(by part (iii)).} \end{aligned}$$

Similarly, $\text{Ann}_V(T \cap U) = \text{Ann}_V(T) + \text{Ann}_V(U)$ is proved.

□

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- [1] M. Bhat, S. Dubey, S. K. Masuti, T. Okuma, J. K. Verma, K. Watanabe, and K. Yoshida. “On Gorensteiness of associated graded rings of filtrations”. *Communications in Algebra*, 1–21. <https://doi.org/10.1080/00927872.2026.2619454> (2026).
- [2] M. Bhat, S. Dubey, S. K. Masuti. “Symmetric decomposition of the Hilbert function of an ideal”. [arXiv:202503.21173](https://arxiv.org/abs/202503.21173) (2025).
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