INDO-EUROPEAN CONFERENCE ON MATHEMATICS 2026

WARING RANK OF BINOMIALS WHOSE ANNIHILATOR IS A COMPLETE INTERSECTION

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WARING RANK OF FORMS - WHAT IS THIS ABOUT?

Let $S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{i \geq 0} S_i$ be the standard graded polynomial ring, and $F \in S$ be a homogeneous polynomial of degree d, also known as a degree d form. It is well known that there exist $L_1, \ldots, L_r \in S_1$ such that

$$F = \sum_{i=1}^{r} \alpha_i L_i^d,$$

for some $\alpha_i \in \mathbb{C}$, $1 \leq i \leq r$ (see [5]). Such a decomposition of F is called *Waring decomposition* or *sum of powers decomposition*. The least integer r such that there exists a Waring decomposition of F with exactly r summand is called the *Waring rank* of F, $\operatorname{rk} F$.

Question: Given $F \in S_d$, what is the Waring rank of F?

Determining or even bounding the Waring rank of a given form is a fundamental but notoriously difficult problem in algebraic geometry and computational algebra. One powerful tool for studying Waring rank is apolarity theory.

Let $T = \mathbb{C}[X_0, X_1, \ldots, X_n]$ be the ring of differential operators with the natural action on S defined by $X_i = \frac{\partial}{\partial x_i}$. For a homogeneous form F, the annihilator ideal of F is defined as $F^\perp = \{g \in T \mid g \circ F = 0\}$. This is a homogeneous Artinian Gorenstein ideal. A set of points $\mathbb{X} \subseteq \mathbb{P}^n$ is said to be **apolar** to F if the defining ideal of $\mathbb{X}$, $I(\mathbb{X}) \subset F^\perp$.

SETUP

We compute the Waring rank of forms $F = m_1 + m_2$ such that $F^\perp$ is a complete intersection (CI) and m_1, m_2 are monomials. In recent works [1, 4, 6] the authors characterized a binomial F such that the corresponding apolar algebra $T/F^\perp$ where $T = \mathbb{C}[X_0, \ldots, X_n]$ is a complete intersection. In fact, they showed that the annihilator ideal $F^\perp$ of such binomials is a *quasi-monomial complete intersection* in “most" of the cases, that is, an ideal of the form $F^\perp = (X_0^{a_0} + G,\ X_1^{a_1},\ X_2^{a_2},\ \ldots,\ X_n^{a_n})$ where G is a homogeneous form in T of degree a_0 with $\deg_{X_0} G < a_0$. Motivated by this, more generally we compute the Waring rank of forms whose annihilator is a quasi-monomial CI.

WARING RANK OF QUASI-MONOMIAL CI

Theorem 1. Let $F \in T$ such that $F^\perp = (X_0^{a_0} + G,\ X_1^{a_1},\ X_2^{a_2},\ \ldots,\ X_n^{a_n})$ where G is a form of degree a_0 with $\deg_{X_0} G < a_0$ and $a_1 \leq a_2 \leq \cdots \leq a_n$. Then

$$\operatorname{rk} F = \begin{cases} a_0 \prod_{j=2}^{n} a_j & \text{if } a_1 \leq a_0 \text{ or} \\ & a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ does not have a multiple factor.} \\ \prod_{j=1}^{n} a_j & \text{if } a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ has a multiple factor.} \end{cases}$$

BINOMIALS IN TWO VARIABLES

Theorem 2[6]. Let $S = \mathbb{C}[x_0, x_1]$, and $T = \mathbb{C}[X_0, X_1]$ be the standard graded polynomial rings. Let

$$F = x_0^{a_0} x_1^{a_1} (x_0^{b_0} - x_1^{b_0})$$

where $b_0, a_0, a_1 \in \mathbb{N}$ with $b_0 \geq 1$, $0 \leq a_0 \leq a_1$. Let

$$v = \min\{i \mid a_0 + 1 \leq i b_0\}, \quad w = \min\{i \mid a_1 + 1 \leq i b_0\}.$$

Then

$$F^\perp = \begin{cases} \left(X_0^{a_0+b_0+1},\ M = \sum_{i=0}^{v} X_0^{ib_0} X_1^{a_1+1-ib_0} \right) & \text{if } v < w; \\ \left(P = \sum_{i=0}^{v} X_0^{ib_0} X_1^{(v-i)b_0},\ Q = \sum_{i=0}^{v-1} X_0^{a_0+1-(v-1-i)b_0} X_1^{a_1+1-ib_0} \right) & \text{if } v = w. \end{cases}$$

BINOMIALS WITH ANNIHILATOR IDEAL IS CI

Theorem 3([1, Theorem 4.1], [4, Theorem 3.3], [6, Theorem 1.3]) Let

$$F = X_0^{a_0} \cdots X_k^{a_k} (X_0^{b_0} - X_1^{b_1} \cdots X_k^{b_k})$$

be a binomial form with all $b_i \neq 0$ such that $\sum_{i=1}^{n} b_i = b_0$ and $k \geq 2$. If there exists an integer $1 \leq i \leq k$ with $a_i < q b_i$ where $q = \lfloor \frac{a_0+1}{b_0} \rfloor$, then

$$F^\perp = (X_0^{a_0+1} + G, X_1^{a_1+b_1+1}, \ldots, X_k^{a_k+b_k+1})$$

where $G = \sum_{j=1}^{m} (X_1^{b_1} \cdots X_k^{b_k})^j X_0^{a_0+1-jb_0}$ with $m = \min\{\lfloor \frac{a_i}{b_i} \rfloor \mid 1 \leq i \leq k\}$.

MAIN THEOREMS

Theorem 4 [2]. Let $F = F_1 F_2$ where $F_1 = x_0^{a_0} x_1^{a_1} (x_0^{b_0} - x_1^{b_0})$, and $F_2 = x_2^{a_2} \cdots x_n^{a_n}$ such that $a_0 \leq a_1$, and $a_2 \leq \cdots \leq a_n$. Let $\lambda = \min\{a_2 + 1, a_0 + b_0 + 1\}$. Using the notations as in Theorem 2 if $v < w$, then

$$\operatorname{rk} F = \begin{cases} \frac{(a_0 + b_0 + 1)}{\lambda} \prod_{i=1}^{n} (a_i + 1) & \text{if } \lambda \leq a_1 + 1 \text{ or } (a_1 + 1 < \lambda \text{ and } a_1 + 1 - v b_0 \leq 1) \\ (a_0 + b_0 + 1) \prod_{i=2}^{n} (a_i + 1) & \text{otherwise.} \end{cases}$$

If $v = w$, then we have the following cases:

$$\operatorname{rk} F = \begin{cases} \deg P \deg Q \left(\prod_{i=3}^{n} (a_i + 1) \right) & \text{if one of the following holds:} \\ & (i)\ a_2 + 1 \leq \min\{\deg P, \deg Q\} \\ & (ii)\ \deg P \leq a_2 + 1 \leq \deg Q \\ & (iii)\ \deg Q \leq a_2 + 1 \leq \deg P \text{ and } Q \text{ is square-free} \\ \operatorname{rk} F_1 \left(\prod_{i=2}^{n} (a_i + 1) \right) & \text{otherwise.} \end{cases}$$

Theorem 5[2]. Let $F = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} (x_0^{b_0} - x_1^{b_1} \cdots x_n^{b_n})$. Write $F = F_1 F_2$ where $F_1 = x_0^{a_0} x_1^{a_1} \cdots x_k^{a_k} (x_0^{b_0} - x_1^{b_1} \cdots x_k^{b_k})$ a binomial with $b_i \neq 0$ for $1 \leq i \leq k$, $b_{k+1} = \cdots = b_n = 0$, and $F_2 = x_{k+1}^{a_{k+1}} \cdots x_n^{a_n}$ for some $1 \leq k \leq n$, and $a_{k+1} \leq \cdots \leq a_n$.

If $2 \leq k \leq n$, and F_1 satisfies conditions as in Theorem , then

$$rk(F) = \begin{cases} \frac{(a_0 + 1)}{\lambda} \prod_{i=1}^{k} (a_i + b_i + 1) \prod_{i=m+1}^{n} (a_i + 1) & \text{if } \lambda \leq a_0 + 1 \\ \prod_{i=1}^{k} (a_i + b_i + 1) \prod_{i=k+1}^{n} (a_i + 1) & \text{otherwise} \end{cases}$$

where $\lambda = \min(\{a_i + b_i + 1 \mid 1 \leq i \leq k\} \cup \{a_{k+1} + 1\})$.

NOTE

We use Bertini’s theorem and the Apolarity lemma to get an upper bound for the Waring rank. We obtain a lower bound using e-computability [3].

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